Theorems on Partial Fractions 5/10/99

Theorem 1

If $a \neq b$ then

$$\frac{1}{(s-a)^m (s-b)^n} = \sum_{i=1}^{m} \frac{(-1)^{m-i}\binom{(n+m-1)-i}{n-1}}{(a-b)^{n+m-i}\,(s-a)^i} + \sum_{j=1}^{n} \frac{(-1)^{m}\binom{(n+m-1)-j}{m-1}}{(a-b)^{n+m-j}\,(s-b)^j}$$

Example: m = 2, n = 3

$$\frac{1}{(s-a)^2 (s-b)^3} = \sum_{i=1}^{2} \frac{(-1)^{2-i}\binom{4-i}{2}}{(a-b)^{5-i}\,(s-a)^i} + \sum_{j=1}^{3} \frac{\binom{4-j}{1}}{(a-b)^{5-j}\,(s-b)^j}$$

$$= \frac{-\binom{3}{2}}{(a-b)^4 (s-a)} \quad \frac{+\binom{2}{2}}{(a-b)^3 (s-a)^2} \quad \frac{+\binom{3}{1}}{(a-b)^4 (s-b)} \quad \frac{+\binom{2}{1}}{(a-b)^3 (s-b)^2} \quad \frac{+\binom{1}{1}}{(a-b)^2 (s-b)^3}$$

$$= -\frac{\frac{3}{(a-b)^4}}{s-a} + \frac{\frac{1}{(a-b)^3}}{(s-a)^2} + \frac{\frac{3}{(a-b)^4}}{s-b} + \frac{\frac{2}{(a-b)^3}}{(s-b)^2} + \frac{\frac{1}{(a-b)^2}}{(s-a)^3}$$

Theorem 2

If $a_i \neq a_j$ then

$$\frac{1}{(x+a_1)\ (x+a_2)\cdots(x+a_n)} = \sum_{i=1}^{n} \frac{(-1)^{n-1}}{\prod_{j=1}^{i-1}(a_i - a_j) \cdot \prod_{j=i+1}^{n}(a_i - a_j)\cdot(x+a_j)}$$

Example: $n = 3$

$$\frac{1}{(x+a_1)(x+a_2)(x+a_3)} = \frac{1}{(a_1-a_2)(a_1-a_3)(x+a_1)} + \frac{1}{(a_2-a_1)(a_2-a_3)(x+a_2)} + \frac{1}{(a_3-a_1)(a_3-a_2)(x+a_3)}$$