Theorem 1:
An irrational number times a rational number equals an irrational number.
Proof (by contradiction):
Let i = irrational, and let p/q be a rational where p and q are integers.
Assume $i \cdot \frac{p}{q} = \frac{a}{b}$ where a and b are integers $\Rightarrow i = \frac{a \cdot q}{b \cdot p} \Rightarrow i = \text{rational}$ //

Theorem 2:
An irrational number plus a rational number equals an irrational number.
Proof (by contradiction):
Let i = irrational, and let p/q be a rational where p and q are integers.
Assume $i + \frac{p}{q} = \frac{a}{b}$ where a and b are integers $\Rightarrow i = \frac{a}{b} - \frac{p}{q} = \frac{aq - bp}{bq} \Rightarrow i = \text{rational}$ //

Theorem 3:
Given any irrational nummber i and any ε, there exist a rational number p/q such that $|i - p/q| < \varepsilon$.
In other words, any irrational number can be approximated with a rational number as close as we want.
Proof

Let $\varepsilon = 1/10^n$ where n is an integer. $|i - p/q| < 1/10^n \Rightarrow i - p/q < 1/10^n$ and $p/q - i < 1/10^n \Rightarrow$

$i - 1/10^n < p/q < i + 1/10^n \Rightarrow i \cdot q - q/10^n < p < i \cdot q + q/10^n$

Let $q = 1/\varepsilon = 10^n \Rightarrow i \cdot 10^n - 1 < p < i \cdot 10^n + 1$ (1)

Claim $p = \text{int}(i \cdot 10^n) + 1$ which will satify the above inequality.

$i \cdot 10^n$ is irrational by above theorem $\Rightarrow i \cdot 10^n \neq \text{integer} \Rightarrow \text{int}(i \cdot 10^n) < i \cdot 10^n \Rightarrow$

$p = \text{int}(i \cdot 10^n) + 1 < i \cdot 10^n + 1$ satisfying right side of (1)

Let z be any number. Then by definition of int(), $z - \text{int}(z) < 1 \Rightarrow z - \text{int}(z) < 2 \Rightarrow i \cdot 10^n - \text{int}(i \cdot 10^n) < 2$

$\Rightarrow i \cdot 10^n < \text{int}(i \cdot 10^n) + 2 \Rightarrow i \cdot 10^n - 1 < p = \text{int}(i \cdot 10^n) + 1$ satisfying left side of (1) //

Example:
Let $i = \sqrt{2}$ and let $\varepsilon = 1/10^9$. Then $q = 1/\varepsilon = 10^9$ and $p = \text{int}(i \cdot 10^9) + 1 = 1414213562 + 1 = 1414213563$

$\Rightarrow p/q = \frac{1414213563}{1000000000} = 1.424213563$

Check:
$|\sqrt{2} - \frac{1414213563}{1000000000}| = |1.4142135623731\cdots - 1.414213563| = 0.000000000626\cdots < \varepsilon$ //

Simply stated, take the decimal representation of the irrational number to n places, and add a one to the least significant digit.

Theorem 4:

Let p be a prime and let m and n be integers such that n does not divide m, then $p^{m/n}$ is irrational

Proof (by contradiction):

n does not divide m $\Rightarrow$ $m/n = c + d/n$ where c and d are integers, $c \geq 0$, and $d < n$ $\Rightarrow$ $p^{m/n} = p^{c+d/n}$ $\Rightarrow$ $p^{m/n} = p^{c+d/n} = p^c \cdot p^{d/n}$. Now p^c is rational $\Rightarrow$ $p^c \cdot p^{d/n}$ and $p^{d/n}$ share the same rationality because dividing a rational by a rational yields a rational, and dividing an irrational by a rational yields an irrational.

So the problem becomes somewhat simplified to that of testing the rationality of $p^{d/n}$.

Assume $p^{d/n} = a/b$ where a and b are positive integers reduced to lowest terms $\Rightarrow$ $p^d = a^n/b^n$ $\Rightarrow$ $a^n = p^d b^n$ $\Rightarrow$ p divides a $\Rightarrow$ $a = pr$, r an integer, $\Rightarrow (pr)^n = p^d b^n$ $\Rightarrow$ $p^n r^n = p^d b^n$ $\Rightarrow$ $p^{n-d} r^n = b^n$ $\Rightarrow$ p divides b contradicting the fact that a and b were reduced to lowest terms. Therefore $p^{d/n}$ is irrational $\Rightarrow$ $p^c \cdot p^{d/n} = p^{m/n}$ is irrational //

Theorem 5:

Let p and q be primes and let m and n be integers such that n does not divide m, then $(pq)^{m/n}$ is irrational

Proof (by contradiction):

n does not divide m $\Rightarrow$ $m/n = c + d/n$ where c and d are integers, $c \geq 0$, and $d < n$ $\Rightarrow$ $(pq)^{m/n} = (pq)^{c+d/n}$ $\Rightarrow$ $(pq)^{m/n} = (pq)^{c+d/n} = (pq)^c \cdot (pq)^{d/n}$. Now $(pq)^c$ is rational $\Rightarrow$ $(pq)^c \cdot (pq)^{d/n}$ and $(pq)^{d/n}$ share the same rationality because dividing a rational by a rational yields a rational, and dividing an irrational by a rational yields an irrational. So the problem becomes somewhat simplified to that of testing the rationality of $(pq)^{d/n}$.

Assume $(pq)^{d/n} = a/b$ where a and b are positive integers reduced to lowest terms $\Rightarrow$ $(pq)^d = a^n/b^n$ $\Rightarrow$ $a^n = (pq)^d b^n$ $\Rightarrow$ pq divides a $\Rightarrow$ $a = pqr$, r an integer, $\Rightarrow (pqr)^n = (pq)^d b^n$ $\Rightarrow$ $(pq)^n r^n = (pq)^d b^n$ $\Rightarrow$ $(pq)^{n-d} r^n = b^n$ $\Rightarrow$ pq divides b contradicting the fact that a and b were reduced to lowest terms. Therefore $(pq)^{d/n}$ is irrational $\Rightarrow$ $(pq)^c \cdot (pq)^{d/n} = (pq)^{m/n}$ is irrational //

Corollary:

Let $p_1, p_2, \cdots p_r$ be r distinct primes, and let m and n be integers such that n does not divide m.

Then $(p_1 \cdot p_2, \cdots p_r)^{m/n}$ is irrational

Theorem 6:
Let z be an integer, and let m and n be integers such that n does not divide m.

Then $z^{m/n}$ is either an integer or it is irrational.

Proof

Case 1: $z^{m/n}$ is an integer. Example $8^{1/3}$ is an integer.

Case2 : (Proof by contradiction) $z^{m/n}$ is a non - integer and assumed rational.

n does not divide m $\Rightarrow$ $m/n = c + d/n$ where c and d are integers, $c \geq 0$, and $d < n \Rightarrow z^{m/n} = z^{c+d/n}$

$\Rightarrow z^{m/n} = z^{c+d/n} = z^c \cdot z^{d/n}$. Now z^c is rational integer $\Rightarrow z^c \cdot z^{d/n}$ and $z^{d/n}$ share the same rationality because dividing a rational by a rational yields a rational, and dividing an irrational by a rational yields an irrational. So the problem becomes somewhat simplified to that of testing the rationality of $z^{d/n}$.

$z^{d/n} = z^{m/n} / z^c \Rightarrow z^{d/n}$ is a non - integer because a non - integer divided by an integer is a non - integer.

Assume $z^{d/n} = a/b$ where a and b are positive integers such that b does not divide a since it is assumed that $z^{d/n}$ is a non - integer $\Rightarrow z^d = a^n/b^n$. But here lies the contradiction. The left side z^d is an integer, but the right side a^n/b^n is a non - integer because if b does not divide a, then b^n will not divide a^n.

Therefore if $z^{d/n}$ is a non - integer, then it must be irrational $\Rightarrow z^c \cdot z^{d/n} = z^{m/n}$ is irrational//

Axiom:
Let c be real such that c is an arbitrary constant, and let n be a positive integer.

Then $\lim_{n \to \infty} c/n = 0$

Lemma 1:
Let c be real such that c is an arbitrary constant, and let n be a positive integer.

Then $\lim_{n \to \infty} n + c = n$

Proof

$\lim_{n \to \infty} c/n = 0 \Leftrightarrow \lim_{n \to \infty} 1 + c/n = 1 \Leftrightarrow \lim_{n \to \infty} n + c = n$ //

In other words, a finite number added to an infinite number leaves the infinite number unchanged.

Lemma 2:

Let r and z be real such that $r > 1$ and $z > 0$ then $r^z > 1$

Proof

$r > 1 \Rightarrow \ln r > 0 \Rightarrow z \cdot \ln r > 0 \Rightarrow e^{z \cdot \ln r} > e^0 \Rightarrow r^z > 1$//

Lemma 3: (To be completed)
Let s, c be real such that $s > 1$, and c an arbitrary constant, and let n be a positive integer.

Then $\lim_{n \to \infty} s^n + c = s^n$

Proof