**Motivation (Objective):**
This is an attempt to answer some questions about the total number of integers, rationals, and irrationals, and how they compare with one another in the real number system. Is the number of irrationals greater than the number of rationals, and if so, by how much? Is the number of rationals greater than the number of integers, and if so, by how much? Do different levels of infinity exist?
To answer these questions some definitions and axioms based on this writer's intuition were utilized. I understand that they may be of concern and even raise the eyebrows of some other mathematicians. However, after assuming the given definitions and axioms are true, hopefully the theorems that follow are correct based on sound logic. In any event, I am sure the results will be found as interesting.

Added Note:
I believe this work just scratches the surface to other theories and problems involving infinite quantities.

The author

**Definition: 1** (N)
Let N = cardinality of {0, 1, 2, · · · }i.e. the cardinality of the set of whole numbers.

**Axiom 1:**
Let i be any integer, then N + i = N. If r be a non-integer, then N + r is undefined.

**Definition: 2** (p/q · N)
Let p/q be a rational with integers p and q, then p/q · N = cardinality of
{0, 0, 0, · · · 0, 0, 0, q, q, q, · · · q, q, q, 2q, 2q, 2q, · · · 2q, 2q, 2q, 3q, 3q, 3q, · · · 3q, 3q, 3q, · · · }
where the number (amount) of zeroes = the number of qs = the number of 2qs = the number of 3qs etc. = p.

**Examples:**
1) 3 · N = cardinality of {0, 0, 0, 1, 1, 1, 2, 2, 2, 3, 3, 3, 4, 4, 4 · · ·}.
2) N/7 = 1/7 · N = cardinality of {0, 7, 14, 21, 28, · · ·}.
3) 3/7 · N = cardinality of {0, 0, 0, 7, 7, 7, 14, 14, 14, 21, 21, 21, 28, 28, 28 · · ·}.
Note: r · N is undefined for r irrational.

**Axiom 2:**
Let a and b be rational such that a < b, then $a \cdot N < b \cdot N$ and $a \cdot N + b \cdot N = (a + b) \cdot N$

**Definition: 3** ($N^k$)
Let k be any positive integer and let S = cardinality of the set of all k tuples of whole numbers. Then the cardinality of S = $N^k$. Note: $N^k$ is undefined for non-integer k.

**Example:**
Let S = cardinality of {(0,0), (0,1), (0,2), · · · (1,0), (1,1), (1,2), · · · (2,0), (2,1), (2,2), · · · }i.e. the cardinality of the set all ordered pairs of whole numbers. Then S = $N^2$. In other words we are assuming the existence of different levels of infinity.

**Axiom 3:**
Let a and b be positive integers such that a < b, then $N^a < N^b$ and $N^a \cdot N^b = N^{a+b}$

**Note:** From Axioms 1, 2, and 3 it is observed that adding or subtracting a finite number to an infinite quantity has no effect on the infinite quantity. However, multiplying, dividing, or raising to a power an infinite quantity by a finite number does have an effect implying the existence of different degrees of infinity.

**Definition: 4** ($r^N$)
r is a positive real, then $r^N$ is defined as r multiplied by itself N times i.e. $r^N = r \cdot r \cdot r \cdot r \cdot r \cdots$

**Theorem:** ( $r \cdot 10^N = 10^N$)
Let r be any positive real number, then $r \cdot 10^N = 10^N$
Proof:
r real => there exist integer n such that $10^{n-1} \leq r \leq 10^n$ => $10^{n-1} \cdot 10^N \leq r \cdot 10^N \leq 10^n \cdot 10^N$ =>
$10^{N+n-1} \leq r \cdot 10^N \leq 10^{N+n}$ Then by Axiom 1, $10^N \leq r \cdot 10^N \leq 10^N$ => $r \cdot 10^N = 10^N$ //

**Axiom 4:**
Any finite real number r can be expressed in decimal form with a finite number of digits before the decimal point, and N digits after the decimal point in the following way:
r = xxx · · · xxx **.** xxxxxx · · · where x is any digit 0 thru 9 and the number of digits before the decimal xxx · · · xxx is finite, and the number of digits after the decimal xxxxxx · · · is N where N = cardinality of {0, 1, 2, · · · }.

**Theorem 1:**
If R = total number of reals, then $\mathbf{R = 10^N}$.
Proof:
Let r be any finite real number, and let y = number of digits before the decimal point. Now the number of digits after the decimal is N by Axiom 4. Therefore since there are 10 digits in the decimal system, the total number of finite real numbers $R = 10^y \cdot 10^N \Rightarrow R = 10^{y+N}$ and by Axiom 1, $R = 10^{y+N} = 10^N$

**Theorem 2:** (Ratio Ir/Ra)
Let N = cardinality of {0, 1, 2, · · · }. In the real number system, let Ra = number of rationals, and let Ir = number of irrationals, then $\mathbf{Ir / Ra = 10^{N/2}}$
Note: This theorem is not restricted to the finite real number system. The Ir/Ra ratio will be the same if the entire real number system was treated. However, for the sake of clarity, this theorem compares the finite rationals with finite irrationals.
**Proof**
Now any rational number q can be expressed as a terminating or repeating decimal and by Axiom 4 will have one of the following forms:

‘ y digits ‘ ‘ z digits ‘ ‘ N−z digits ‘
$q_1$ = xxx · · · xxx **.** xxx · · · xxx a a a a a a a · · · or
$q_2$ = xxx · · · xxx **.** xxx · · · xxx ab ab ab ab · · · or
$q_3$ = xxx · · · xxx **.** xxx · · · xxx abc abc abc · · · or
$q_4$ = xxx · · · xxx **.** xxx · · · xxx abcd abcd abcd · · ·
etc.

where x, a, b, c, d, are digits 0 thru 9, the number of digits in front of the decimal = y (finite), and z non-repeating digits to the right of the decimal, followed by N−z = N (infinite) digits with a repeating pattern.
Notes:
1. Any finite rational number q can be expressed using y + z + N = N digits by Axiom 4.
2. There are no pattern restrictions on the first y digits before the decimal or the first z digits after the decimal.
3. The N−z repeating digits may all be zeroes thus accounting for all terminating decimals.
4. The number of finite rationals by type are as follows:
   type $q_1 = 10^y \cdot 10^z \cdot 10 = 10^{y+z+1}$
   type $q_2 = 10^y \cdot 10^z \cdot 10^2 = 10^{y+z+2}$
   type $q_n = 10^y \cdot 10^z \cdot 10^n = 10^{y+z+n}$ where n is the number of digits repeated.
   Examples:
   type $q_1$: 5.123466666666 · · ·, 37.22222222 · · · , 54.00000000 · · ·
   type $q_2$: 5.123467676767 · · ·, 37.5823232323 · · · , 54.15151515 · · ·
   type $q_3$: 5.1234678678678 · · ·, 37.45234234234 · · · , 54.152152152 · · ·

Number of rationals of type $q_1$ = 1000 · · · 000 (1 followed by y + z + 1 zeroes)
Number of rationals of type $q_2$ = 10000 · · · 000 (1 followed by y + z + 2 zeroes)
Number of rationals of type $q_n$ = 10000000 · · · 000 (1 followed by y + z + n zeroes) etc.

**5.** Let Ra = total number of rationals, then $R_a = \sum_{n=1}^{\infty} q_n$

It is interesting to note that Ra is made up of n ones followed by y + z + 1 zeroes i.e.

$$Ra = \text{limit as } n \to \infty \; q_1 + q_2 + \cdots + q_n = \underbrace{111 \cdots 111}_{\text{n ones}} \underbrace{00000 \cdots 00000}_{\text{y + z + 1 zeroes}}$$

The question becomes what is the relationship between n and N as n –> ∞. Consider the following chart neglecting any digits to the left of the decimal where n is the number of digits repeated, and D is the number of digits behind the decimal.

| **Number** | **n** (# of digits repeated) | **D** (# of digits to right of decimal) | **n/D** |
|---|---|---|---|
| 0.a | 0 | 1 | 0 |
| 0.aa | 1 | 2 | 1/2 |
| 0.aaa | 1 | 3 | 1/3 |
| 0.ab | 0 | 2 | 0 |
| 0.abab | 2 | 4 | 1/2 |
| 0.ababab | 2 | 6 | 1/3 |
| 0.abc | 0 | 3 | 0 |
| 0.abcabc | 3 | 6 | 1/2 |
| 0.abcabcabc | 3 | 9 | 1/3 |
| 0.abcabcabcabc | 3 | 12 | 1/4 |
| 0.abcd | 0 | 4 | 0 |
| 0.abcdabcd | 4 | 8 | 1/2 |
| 0.abcdabcdabcd | 4 | 12 | 1/3 |
| 0.abcde | 0 | 5 | 0 |
| 0.abcdeabcde | 5 | 10 | 1/2 |
| 0.abcdeabcdeabcde | 5 | 15 | 1/3 |
| 0.abcdef | 0 | 6 | 0 |
| 0.abcdefabcdef | 6 | 12 | 1/2 |
| $0.a_1a_2 \cdots a_na_1a_2 \cdots a_n \cdots$ | n | D | n/D |

The above table reveals that as n –> ∞, and D –> N, n/D ≤ ½ => n ≤ D/2. D ≤ N => n ≤ N/2. In other words n/N can never be larger than 1/2. Choosing the largest n yields

$$R_a = \sum_{i=1}^{N/2} q_i = \sum_{i=1}^{N/2} 10^{y+z+i} \Rightarrow\ I_r = \text{total\# of irrationals} = R - R_a = 10^N - \sum_{i=1}^{N/2} 10^{y+z+i} \Rightarrow$$

$$I_r / R_a = (R - R_a)/R_a = R/R_a - 1 = \frac{10^N}{\sum_{i=1}^{N/2} 10^{y+z+i}} - 1 = \frac{10^N}{10^{y+z}\sum_{i=1}^{N/2} 10^i} - 1 = \frac{10^{N-(y+z)}}{\sum_{i=1}^{N/2} 10^i} - 1 =$$

$$\frac{10^{N-(y+z)}}{\frac{(10^{N/2+1}-1)}{(10-1)}} - 1 = \frac{9 \cdot 10^N}{10^{N/2}} = 9 \cdot 10^{N/2} = 10^{N/2} \text{ by a previous theorem } //$$

**Corollary 1:** (Ratio Ir / R)

$$I_r / R = I_r / 10^N = \left(10^N - \sum_{i=1}^{N/2} 10^{y+z+i}\right) / 10^N = 1 - \sum_{i=1}^{N/2} 10^{y+z+i-N} = 1 - 10^{y+z-N} \sum_{i=1}^{N/2} 10^i =$$

$$1 - 10^{y+z+1-N} \sum_{i=0}^{N/2-1} 10^i = 1 - 10^{y+z+1-N} \frac{10^{N/2} - 1}{10 - 1} = 1 - 10^{-N} \frac{10^{N/2}}{9} = 1 - \frac{10^{N/2}}{9 \cdot 10^N} =$$

$$1 - \frac{1}{9 \cdot 10^{N/2}} = 1 - \frac{1}{10^{N/2}} = 1 - (0) = 1 \ //$$

**Corollary 2:** (Ratio Ra /R)

$$R_a / R = \sum_{i=1}^{N/2} 10^{y+z+i} / 10^N = \sum_{i=1}^{N/2} 10^{y+z+i-N} = 10^{y+z-N} \sum_{i=1}^{N/2} 10^i =$$

$$10^{y+z+1-N} \sum_{i=0}^{N/2-1} 10^i = 10^{y+z+1-N} \frac{10^{N/2} - 1}{10 - 1} = 10^{-N} \frac{10^{N/2}}{9} = \frac{10^{N/2}}{9 \cdot 10^N} = \frac{1}{9 \cdot 10^{N/2}} = \frac{1}{10^{N/2}} = 0 \ //$$

**Theorem 3:**(Ratio I /Ra)
Let N = order of {0, 1, 2, · · · }i.e. the order of the set of whole numbers. Let Ra = total number of rationals, and I = total number of integers, then the ratio of Integers to Rationals = **I /Ra = 0**

**Proof**
If N = order of {0, 1, 2, · · · } then I = 2N = order of {· · · −2, −1, 0, 1, 2, · · · }the set of all integers. By Theorem 2, Ra = $10^{y+z+1} + 10^{y+z+2} + \cdots + 10^{N/2}$ => $10^{N/2} \le$ Ra => $2N / Ra \le 2N / 10^{N/2}$. Now the limit as n –> N of $2n / 10^{n/2} = 0$ => 2N / Ra = I / Ra ≤ 0 and since I and Ra are both positive, I / Ra = 0 //

**Notes:**
1. Theorem 2 reveals that the total number of rationals (an infinite set) is insignificant to the total number of Irrationals, and Theorem 3 reveals that the total number of integers is insignificant to the total number of rationals supporting the "different levels of infinity" concept.
2. It remains to be shown how the total number of transcendentals compare with I, Ra, Ir, and R.

**Another Approach**

**Theorem:**

$R_a = 10^{N/2}$

Proof :

$$R_a = \sum_{i=1}^{N/2} 10^{y+z+i} = 10^{y+z} \sum_{i=1}^{N/2} 10^i = 10^{y+z+1} \sum_{i=0}^{N/2-1} 10^i = 10^{y+z+1} \frac{10^{N/2}-1}{10-1} =$$

$$10^{y+z+1} \frac{10^{N/2}}{9} = \frac{10^{N/2+y+z+1}}{9} = \frac{1}{9} 10^{N/2} = 10^{N/2} \quad //$$

**Lemma:**

Given c real, $c > 1$, and a and b rational with $a < b$, then $\lim_{n \to \infty} c^{bn} \pm c^{an} = c^{bn}$

Proof :

$$\text{Consider } \lim_{n \to \infty} \frac{c^{bn} \pm c^{an}}{c^{bn}} = \lim_{n \to \infty} 1 \pm \frac{c^{an}}{c^{bn}} = 1 \pm \frac{1}{c^{(b-a)n}} = 1 \pm 0 = 1$$

$$\text{Now } \lim_{n \to \infty} \frac{c^{bn} \pm c^{an}}{c^{bn}} = 1 \Rightarrow \lim_{n \to \infty} c^{bn} \pm c^{an} = c^{bn} \quad //$$

**Theorem:**

$I_r = 10^N$

Proof :

$$I_r = R - R_a = 10^N - \sum_{i=1}^{N/2} 10^{y+z+i} = 10^N - 10^{y+z} \sum_{i=1}^{N/2} 10^i = 10^N - 10^{y+z+1} \sum_{i=0}^{N/2-1} 10^i =$$

$$10^N - 10^{y+z+1} \frac{10^{N/2}-1}{10-1} = 10^N - 10^{y+z+1} \frac{10^{N/2}}{9} = 10^N - \frac{10^{N/2+y+z+1}}{9} =$$

$$10^N - \frac{1}{9} 10^{N/2} = 10^N - 10^{N/2} = 10^N \text{ by above lemma } //$$

**Theorem:**

$I_r / R_a = 10^{N/2}$

Proof :

$$I_r / R_a = 10^N / 10^{N/2} = 10^{N-N/2} = 10^{N/2} \quad //$$