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Binomial Theorem

$$(a+z)^r = a^r + \frac{r}{1!}a^{r-1}z + \frac{r(r-1)}{2!}a^{r-2}z^2 + \frac{r(r-1)(r-2)}{3!}a^{r-3}z^3 + \cdots$$

Proof:

$$f(z) = f(0) + \frac{f^1(0)\cdot z}{1!} + \frac{f^2(0)\cdot z^2}{2!} + \frac{f^3(0)\cdot z^3}{3!} + \cdots \text{ by Taylor Series}$$

$$f(z)=(a+z)^r \Rightarrow f^1(z) = r(a+z)^{r-1},\ f^2(z) = r(r-1)(a+z)^{r-2},\ f^3(z) = r(r-1)(r-2)(a+z)^{r-3}, \text{etc.} \Rightarrow$$

$$f(0)=a^r,\ f^1(0) = r\,a^{r-1},\ f^2(0) = r(r-1)a^{r-2},\ f^3(0) = r(r-1)(r-2)a^{r-3}, \text{etc.} \Rightarrow$$

$$(a+z)^r = a^r + \frac{r}{1!}a^{r-1}z + \frac{r(r-1)}{2!}a^{r-2}z^2 + \frac{r(r-1)(r-2)}{3!}a^{r-3}z^3 + \cdots$$

Example1:

$$f(x) = (a-x^2)^{-\frac{1}{2}} = \frac{1}{\sqrt{a-x^2}} \Rightarrow z = -x^2 \text{ and } r = -1/2 \Rightarrow$$

$$(a-x^2)^{-\frac{1}{2}} = a^{-1/2} + \frac{a^{-3/2}x^2}{2\cdot 1!} + \frac{3\cdot a^{-5/2}x^4}{2^2\cdot 2!} + \frac{3\cdot 5\cdot a^{-7/2}x^6}{2^3\cdot 3!} + \frac{3\cdot 5\cdot 7\cdot a^{-9/2}x^8}{2^4\cdot 4!} + \cdots \Rightarrow$$

$$f(1/2) = (a-1/4)^{-\frac{1}{2}} = a^{-1/2} + \frac{a^{-3/2}}{2^3\cdot 1!} + \frac{3\cdot a^{-5/2}}{2^6\cdot 2!} + \frac{3\cdot 5\cdot a^{-7/2}}{2^9\cdot 3!} + \frac{3\cdot 5\cdot 7\cdot a^{-9/2}}{2^{12}\cdot 4!} + \cdots$$

$$a = 2 \Rightarrow f(1/2) = (7/4)^{-\frac{1}{2}} = 2^{-1/2} + \frac{2^{-3/2}}{2^3\cdot 1!} + \frac{3\cdot 2^{-5/2}}{2^6\cdot 2!} + \frac{3\cdot 5\cdot 2^{-7/2}}{2^9\cdot 3!} + \frac{3\cdot 5\cdot 7\cdot 2^{-9/2}}{2^{12}\cdot 4!} + \cdots \Rightarrow$$

$$(7/4)^{-\frac{1}{2}} = \frac{1}{\sqrt{2}} + \frac{1}{2^4\sqrt{2}\cdot 1!} + \frac{3}{2^8\sqrt{2}\cdot 2!} + \frac{3\cdot 5}{2^{12}\sqrt{2}\cdot 3!} + \frac{3\cdot 5\cdot 7}{2^{16}\sqrt{2}\cdot 4!} + \cdots \Rightarrow$$

$$(7/4)^{-\frac{1}{2}} = \frac{2}{\sqrt{7}} = \frac{1}{\sqrt{2}}\left(1 + \frac{1}{2^4\cdot 1!} + \frac{3}{2^8\cdot 2!} + \frac{3\cdot 5}{2^{12}\cdot 3!} + \frac{3\cdot 5\cdot 7}{2^{16}\cdot 4!} + \cdots\right) //$$

Theorem (Chord to Arc Length)
Given a chord of length C inscribed in a circle of radius one, the arc length S of the arc associated with chord C will be:

$S = \lim_{n \to \infty} S_n$ where S_n is defined as follows:

$$S_1 = 2\sqrt{2-\sqrt{4-C^2}}$$

$$S_2 = 2^2\sqrt{2-\sqrt{2+\sqrt{4-C^2}}}$$

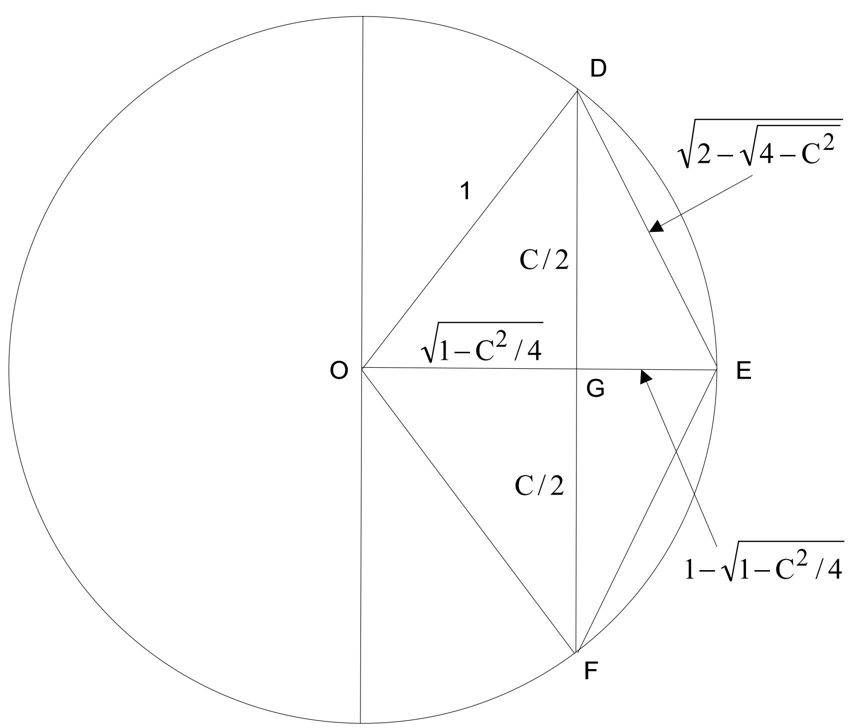
$$S_3 = 2^3\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{4-C^2}}}}$$

$$S_4 = 2^4\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{4-C^2}}}}}$$

$$S_5 = 2^5\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{4-C^2}}}}}} \qquad \text{etc.}$$

Proof:
With respect to Figure (Chord to Arc Length 1), Arc DEF is associated with Chord DF of length C. The objective is to measure arc length of Arc DEF. Note that if Chord C was a diameter (length 2), arc length would equal exactly ½ circumference which equals π.

Figure (Chord to Arc Length 1)



Given OE is perpendicular to chord C $\Rightarrow$ OE bisects chord C $\Rightarrow$

$DG = C/2 \Rightarrow OG = \sqrt{1 - C^2/4} \Rightarrow GE = 1 - \sqrt{1 - C^2/4}$

Now ΔDGE is a right triangle with height C/2 and base GE $\Rightarrow$

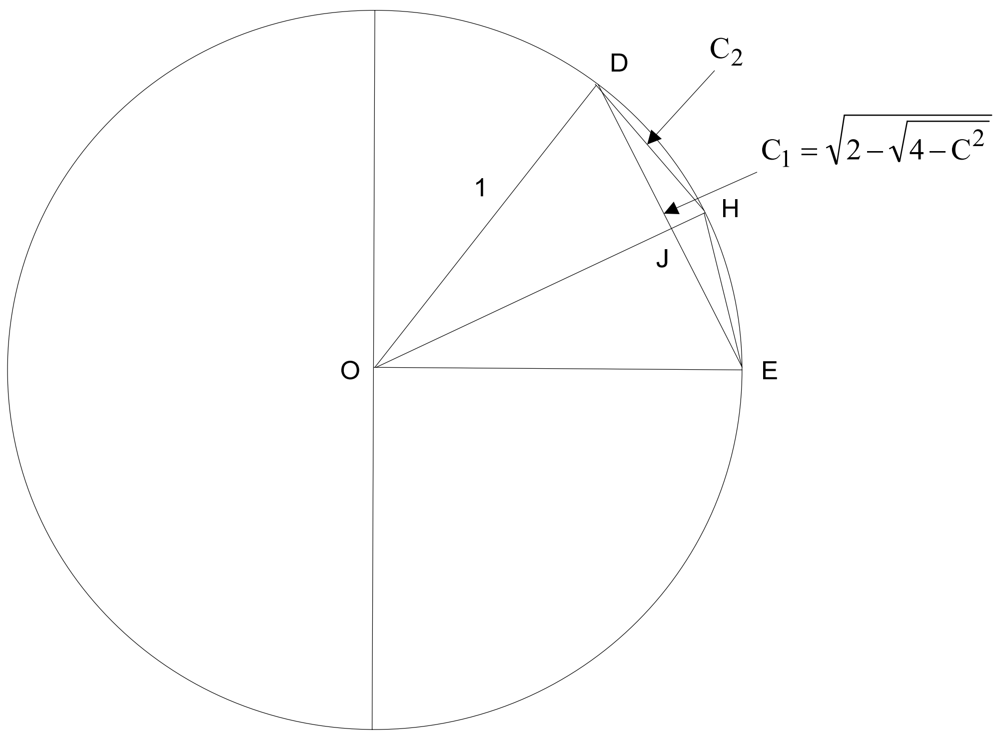
$$DE = \sqrt{(C/2)^2 + \left(1 - \sqrt{1 - C^2/4}\right)^2} = \sqrt{C^2/4 + 1 + \left(1 - C^2/4\right) - 2\sqrt{1 - C^2/4}} = \sqrt{2 - \sqrt{4 - C^2}}$$

Let $C_1 = DE = EF = \sqrt{2 - \sqrt{4 - C^2}}$ and let $S_1 = DE + EF$ be the first approximation to Arc Segment DEF.

and therefore $S_1 = 2\sqrt{2 - \sqrt{4 - C^2}}$

Now draw a radius OH bisecting chord DE and repeat this process to calculate the length of chord C_2 created by connecting point D to point H. See Figure (Chord to Arc Length 2).

Figure (Chord to Arc Length 2)



By examining this process of "chord bisecting" we see that chord C of length C yeilds chord C_1 where $C_1 = \sqrt{2-\sqrt{4-C^2}}$. Therefore repeating this process yeilds $C_2 = \sqrt{2-\sqrt{4-C_1^2}}$,

$C_3 = \sqrt{2-\sqrt{4-C_2^2}}, \cdots, C_n = \sqrt{2-\sqrt{4-C_{n-1}^2}}$

So $C_2 = \sqrt{2-\sqrt{4-C_1^2}} = \sqrt{2-\sqrt{4-\left(\sqrt{2-\sqrt{4-C^2}}\right)^2}} = \sqrt{2-\sqrt{4-\left(2-\sqrt{4-C^2}\right)}} \Rightarrow$

$C_2 = \sqrt{2-\sqrt{2+\sqrt{4-C^2}}}$ From Figures 1 and 2, note that a closer approximation to length of arc DEF is 4 x C_2. Call this S_2 and note $S_2 = 2^2 \cdot \sqrt{2-\sqrt{2+\sqrt{4-C^2}}}$.. Again repeating this process yields

$S_3 = 2^3 \cdot \sqrt{2-\sqrt{2+\sqrt{2+\sqrt{4-C^2}}}}$, $S_4 = 2^4 \cdot \sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{4-C^2}}}}}$ etc.

Therefore arc length of DEF = $\lim_{n \to \infty} S_n$ //

Theorem (maybe useless but interesting byproduct of Chord to Arc Length Theorem)

$2 = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}}}}$

Proof :

Let $n = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}}}} \Rightarrow n^2 = 2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}}} \Rightarrow$

$n^2 = 2 + n \Rightarrow n = 2$ //

Theorem (Arc Length to Chord)
Given an Arc of length S of a circle of radius one, its associated chord will be of length C and $C = \lim_{n \to \infty} C_n$ where C_n is defined as follows:

$$C_1 = \sqrt{4-\left(S^2/4-2\right)^2}$$

$$C_2 = \sqrt{4-\left(\left(S^2/4^2-2\right)^2-2\right)^2}$$

$$C_3 = \sqrt{4-\left(\left(\left(S^2/4^3-2\right)^2-2\right)^2-2\right)^2}$$

$$C_4 = \sqrt{4-\left(\left(\left(\left(S^2/4^4-2\right)^2-2\right)^2-2\right)^2-2\right)^2} \quad \text{etc.}$$

Proof:
Since each S_n in the previous theorem is a function of C, and C and be expressed as an inverse function of S_n, then C_n can be defined for all n as listed below.

$$S_1 = 2\sqrt{2-\sqrt{4-C^2}} \Rightarrow C_1 = \sqrt{4-\left(S^2/4-2\right)^2}$$

$$S_2 = 2^2\sqrt{2-\sqrt{2+\sqrt{4-C^2}}} \Rightarrow C_2 = \sqrt{4-\left(\left(S^2/4^2-2\right)^2-2\right)^2}$$

$$S_3 = 2^3\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{4-C^2}}}} \Rightarrow C_3 = \sqrt{4-\left(\left(\left(S^2/4^3-2\right)^2-2\right)^2-2\right)^2}$$

$$S_4 = 2^4\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{4-C^2}}}}} \Rightarrow C_4 = \sqrt{4-\left(\left(\left(\left(S^2/4^4-2\right)^2-2\right)^2-2\right)^2-2\right)^2} \quad \text{etc.}$$

Therefore $S = \lim_{n \to \infty} S_n \Rightarrow C = \lim_{n \to \infty} C_n$

Theorem (Sum of Reciprocals of Roots)

If $r_1, r_2, \cdots r_n$ are roots to the equation $y = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ then

$a_0 = (-1)^n \cdot r_1 \cdot r_2 \cdots \cdot r_n$ (product of roots), and $-\dfrac{a_1}{a_0} = \dfrac{1}{r_1} + \dfrac{1}{r_2} + \cdots + \dfrac{1}{r_n}$ (sum of reciprocals of roots)

Proof : (by induction)

Case $n=2$: $y = x^2 + a_1x + a_0 = (x-r_1)(x-r_2) = x^2 - (r_1+r_2)x + r_1 \cdot r_2$

Note $\dfrac{1}{r_1} + \dfrac{1}{r_2} = \dfrac{r_1+r_2}{r_1 \cdot r_2} = -\dfrac{a_1}{a_0}$

Case $n=3$: $y = (x-r_1)(x-r_2)(x-r_3) = x^3 - (r_1+r_2+r_3)x^2 + (r_1 \cdot r_2 + r_1 \cdot r_3 + r_2 \cdot r_3)x - r_1 \cdot r_2 \cdot r_3$

Note $\dfrac{1}{r_1} + \dfrac{1}{r_2} + \dfrac{1}{r_3} = \dfrac{r_2 \cdot r_3 + r_1 \cdot r_3 + r_1 \cdot r_2}{r_1 \cdot r_2 \cdot r_3} = -\dfrac{a_1}{a_0}$

Assume true for n $\Rightarrow$ $y = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ and

$$-\frac{a_1}{a_0} = \frac{1}{r_1} + \frac{1}{r_2} + \cdots + \frac{1}{r_n} = \frac{\frac{P}{r_1} + \frac{P}{r_2} + \cdots + \frac{P}{r_n}}{P} \text{ where } P = r_1 \cdot r_2 \cdots \cdot r_n$$

Now $(x-r_{n+1}) \cdot y = (x-r_{n+1}) \cdot (x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0) =$

$(x^{n+1} + a_{n-1}x^n + \cdots + a_1x^2 + a_0x) - r_{n+1}(x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0) =$

$x^{n+1} + b_nx^n + \cdots + b_2x^2 + a_0x - r_{n+1} \cdot a_1x - r_{n+1} \cdot a_0 \Rightarrow$

$b_1 = a_0 - r_{n+1} \cdot a_1$ and $b_0 = -r_{n+1} \cdot a_0 \Rightarrow$

$$-\frac{b_1}{b_0} = \frac{a_0 - r_{n+1} \cdot a_1}{r_{n+1} \cdot a_0} = \frac{1}{r_{n+1}} - \frac{a_1}{a_0} = \frac{1}{r_1} + \frac{1}{r_2} + \cdots + \frac{1}{r_{n+1}}$$

$b_0 = -r_{n+1} \cdot a_0 = (-1)\,r_{n+1}\,(-1)^n \cdot r_1 \cdot r_2 \cdots \cdot r_n = (-1)^{n+1} \cdot r_1 \cdot r_2 \cdots \cdot r_n \cdot r_{n+1}$ //

Theorem (Taylor Series)

If $f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + a_4(x-a)^4 + \cdots \quad (1)$

then $f(x) = f(a) + \frac{f^1(a)(x-a)}{1!} + \frac{f^2(a)(x-a)^2}{2!} + \frac{f^3(a)(x-a)^3}{3!} + \frac{f^4(a)(x-a)^4}{4!} + \cdots$

Proof

$f^1(x) = a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + 4a_4(x-a)^3 + 5a_5(x-a)^4 + \cdots$

$f^2(x) = 2a_2 + 3\cdot 2a_3(x-a) + 4\cdot 3a_4(x-a)^2 + 5\cdot 4a_5(x-a)^3 + \cdots$

$f^3(x) = 3\cdot 2a_3 + 4\cdot 3\cdot 2a_4(x-a) + 5\cdot 4\cdot 3a_5(x-a)^2 + \cdots$

$f^4(x) = 4\cdot 3\cdot 2a_4 + 5\cdot 4\cdot 3\cdot 2a_5(x-a) + \cdots$

$f^5(x) = 5\cdot 4\cdot 3\cdot 2a_5 + \cdots$

etc. $\Rightarrow$

$f(a) = a_0$

$f^1(a) = a_1 \Rightarrow a_1 = \frac{f^1(a)}{1!}$

$f^2(a) = 2a_2 \Rightarrow a_2 = \frac{f^2(a)}{2!}$

$f^3(a) = 3\cdot 2a_3 \Rightarrow a_3 = \frac{f^3(a)}{3!}$

$f^4(a) = 4\cdot 3\cdot 2a_4 \Rightarrow a_4 = \frac{f^4(a)}{4!}$

$f^5(a) = 5\cdot 4\cdot 3\cdot 2a_5 \Rightarrow a_5 = \frac{f^5(a)}{5!}$

etc.

Therefore replacing each a_i in equation (1) with $\frac{f^i(a)}{i!}$ gives the desired result //

Note : Setting $a = 0$ yields the Maclaurin Series

$f(x) = f(0) + \frac{f^1(0)\cdot x}{1!} + \frac{f^2(0)\cdot x^2}{2!} + \frac{f^3(0)\cdot x^3}{3!} + \frac{f^4(0)\cdot x^4}{4!} + \cdots$

Theorem (Sin(x) Series)

$$\mathrm{Sin}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots$$

Proof

$f(x)=\mathrm{Sin}(x),\ f^1(x) = \mathrm{Cos}(x),\ f^2(x) = -\mathrm{Sin}(x),\ f^3(x) = -\mathrm{Cos}(x),\ f^4(x) = \mathrm{Sin}(x),\ f^5(x) = \mathrm{Cos}(x)$ etc.

$\Rightarrow\ f(0) = 0,\ f^1(0) = 1,\ f^2(0) = 0,\ f^3(0) = -1,\ f^4(0) = 0,\ f^5(0) = 1$ etc.

Replacing the $f^i(0)$ in the Maclaurin Series

$$f(x) = f(0) + \frac{f^1(0)(x)}{1!} + \frac{f^2(0)(x)^2}{2!} + \frac{f^3(0)(x)^3}{3!} + \frac{f^4(0)(x)^4}{4!} + \cdots$$

with $0, 1, 0, -1, 0$ etc. gives the desired result

Theorem (e^{ix})

$e^{i\cdot x} = \cos(x) + i\cdot\sin(x)$

Proof:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots$$

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \cdots \text{ from Taylor Series} \Rightarrow$$

$$e^{i\cdot x} = 1 + i\cdot x - \frac{x^2}{2!} - \frac{i\cdot x^3}{3!} + \frac{x^4}{4!} + \frac{i\cdot x^5}{5!} - \frac{x^6}{6!} - \frac{i\cdot x^7}{7!} + \frac{x^8}{8!} + \frac{i\cdot x^9}{9!} + \cdots \Rightarrow$$

$$e^{ix} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots \quad + i\cdot\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots\right)$$