Electronic Component Failure Rate Prediction Analysis

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Introduction:

Many electronic component manufacturers predict failure rates (λ) of their devices by use of an in-house testing process. For example the test may involve placing a certain number of components in an oven, and allowing them to "bake" at a predetermined temperature for a predetermined amount of time. At the end of the test, the number of failed devices are counted to determine the values of r and "DeviceHours", where r = number of failures, and DeviceHours = number of surviving components times the number of hours they remained in the oven. The quotient $\frac{r}{\text{DeviceHours}}$ is considered an estimate of λ as opposed to the true λ. One reason being is that the group of devices under test is just a small sample of a much larger population. Since the manufacturer may have difficulty accumulating enough data to calculate true λ, he does the next best thing. He employs a Chi-Square statistical tool that relates the controlled test sample to the entire population. This tool allows him to calculate a "confidence interval" which provides a "statistical" upper bound for the true λ. The equation manufacturers are using is

$\lambda = \frac{\chi^2(1-\text{CL}, 2r+2)}{2 \text{ x DeviceHours}}$ [eq. 1] where $\chi^2(1-\text{CL}, 2r+2)$ is a statistical factor taken from the Chi-Square Table, with 2r+2 = degrees of freedom, and CL = confidence level.

Objective:

This paper tries to explain the why and how Equation 1 works, and also where it comes from.

Background Theory:

Calculations for confidence level are based on the binomial distribution function described in many statistics textbooks. The binomial distribution function is written as

$$P_n(k) = \binom{n}{k} p^{n-k} q^k \quad \text{where} \binom{n}{k} \text{is defined as} \frac{n!}{k!(n-k)!} \qquad \text{[eq. 2] where } p + q = 1$$

When we are interested in the probability that r or fewer events occur in n trials (or, conversely, that greater than r events occur), then the cumulative binomial distribution function (CDF) of Equation 2 is used as shown in Equations 3 or 4:

$$P(\varepsilon \le r) = \sum_{k=0}^{r} P_n(k) = \sum_{k=0}^{r} \left(\frac{n!}{k!(n-k)!} \right) p^{n-k} q^k \quad \text{[eq. 3]} \quad \text{or}$$

$$P(\varepsilon > r) = 1 - P(\varepsilon \le r) = \sum_{k=r+1}^{n} \left(\frac{n!}{k!(n-k)!} \right) p^{n-k} q^k \quad \text{[eq. 4]}$$

In terms of the cumulative binomial distribution function, the confidence level is defined as

$$CL = P(\varepsilon > r) = 1 - \sum_{k=0}^{r}\left(\frac{n!}{k!(n-k)!}\right)(1-q)^{n-k}q^{k} \quad \text{[eq. 5]} \quad \text{or}$$

$$1 - CL = \sum_{k=0}^{r}\left(\frac{n!}{k!(n-k)!}\right)(1-q)^{n-k}q^{k} \quad \text{[eq. 6]}$$

where CL is the confidence level in percent.

Now if n is very large, and q is very small, then the Poisson expression $\frac{(nq)^k}{k!}e^{-nq}$ provides a conservative estimate of the binomial distribution term for term as shown by Equation 7.

$$\left(\frac{n!}{k!(n-k)!}\right)(1-q)^{n-k}q^{k} \approx \frac{(nq)^k}{k!}e^{-nq} \quad \text{[eq. 7]} \qquad \text{(See attached for a proof.)}$$

Joining Equations 6 and 7 together we get

$$1 - CL = \sum_{k=0}^{r}\frac{(nq)^k}{k!}e^{-nq} = e^{-nq}\left[1 + nq + \cdots + \frac{(nq)^{r-1}}{(r-1)!} + \frac{(nq)^r}{r!}\right] \quad \text{[eq. 8]}$$

where nq is the expected or mean value.

In terms of Reliability, if λ = the constant failure rate of a component, and t = time in operation, then n can be replaced by t, and q replaced by λ, and λt will be the expected (mean) number of failures. This implies that $P(r) = \frac{e^{-\lambda t}\cdot(\lambda t)^r}{r!}$ = probability of exactly r failures in time interval 0 to t. Note that when writing each term out

$$P(0) = e^{-\lambda t},\ P(1) = e^{-\lambda t}\cdot\lambda t,\ P(2) = e^{-\lambda t}\frac{(\lambda t)^2}{2!},\ P(3) = e^{-\lambda t}\frac{(\lambda t)^3}{3!}, \text{etc.}$$

it becomes apparent that $P(r) = e^{-\lambda t}\left[1 + \lambda t + \cdots + \frac{(\lambda t)^{r-1}}{(r-1)!} + \frac{(\lambda t)^r}{r!}\right]$. [eq. 9] In words, Equation 9 reads P(r) = probability (0 failures) or probability (1 failure) or … or probability (r failures). Another way of saying this same thing is the probability of r or less failures in a time interval t. Replacing λt for nq in Equation 8 yields

$$1 - CL = \sum_{k=0}^{r}\frac{(\lambda t)^k}{k!}e^{-\lambda t} = e^{-\lambda t}\left[1 + \lambda t + \cdots + \frac{(\lambda t)^{r-1}}{(r-1)!} + \frac{(\lambda t)^r}{r!}\right] \quad \text{[eq. 10]}$$

It turns out that for a given CL, the term $\chi^2(1-\text{CL},2r+2)/2$ is the exact λt solution to Equation 10. So for example choosing CL = 60% or 1 – CL = 0.4, the following table shows exactly what the entries of the Chi-Square Table represent.

Chi-Square Table:

Entries from Chi-Square Table (Probability, Degrees of Freedom)		
Failures (r)	**(1–CL, 2r+2) / 2 = λt**	**exact solution to**
0	(0.4, 2) / 2 = 0.916290731	$0.4 = e^{-\lambda t}$
1	(0.4, 4) / 2 = 2.022313245	$0.4 = e^{-\lambda t} \cdot [1+\lambda t]$
2	(0.4, 6) / 2 = 3.105378597	$0.4 = e^{-\lambda t} \cdot [1+\lambda t+(\lambda t)^2/2!]$
3	(0.4, 8) / 2 = 4.175262733	$0.4 = e^{-\lambda t} \cdot [1+\lambda t+(\lambda t)^2/2!+(\lambda t)^3/3!]$

In other words $\frac{\chi^2(1-\text{CL},2r+2)}{2} = \lambda t$ in column 2 yields the exact solutions to the Poisson distributions in column 3 of the above table. In other words the Chi Square Table solves for λt. (Note that the Chi Square Table does not solve for λ or t alone, but λt.) Now simply divide both sides of the above equation by t (device hours) to get $\lambda = \frac{\chi^2(1-\text{CL},2r+2)}{2t}$ which is exactly Equation 1. //

Note:
The above method is reserved for devices that exhibit constant failure rates i.e. electronic devices. For devices that exhibit non-constant failure rates i.e. mechanical devices, other statistical methods should be employed which is a subject for another paper.

Notes on Chi-Square Table
A user friendly Chi-Square Interactive Table can be accessed on line at
http://math.uc.edu/~brycw/classes/148/tables.htm

Both Microsoft Excel and MathCad have built in Chi-Square Table generators. Any entry of the Chi-Square Table $\chi^2(1-\mathrm{CL}, 2r+2)$ can be determined using either one of those two programs. For **Excel** enter $\mathrm{Chiinv}(1-\mathrm{CL},\ 2r+2)$. For **MathCad** enter $\mathrm{qchisq}(\mathrm{CL},\ 2r+2)$

More Interesting facts on the Poisson Distribution

The collection of terms $e^{-x},\ e^{-x}x,\ e^{-x}\frac{x^2}{2!},\ e^{-x}\frac{x^3}{3!},\ \cdots$ is called the Poisson Distribution.

Note that the sum of all the terms is equal to one.

$$\sum_{k=0}^{\infty} e^{-x}\frac{x^k}{k!} = e^{-x}\left[1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right] = e^{-x}\cdot e^{x} = 1$$

Therefore this Poisson Distribution is a discrete probability density function or PDF.

The cumulative distribution function CDF is $\sum_{k=0}^{r} e^{-x}\frac{x^k}{k!} = e^{-x}\left[1+x+\cdots+\frac{x^{r-1}}{(r-1)!}+\frac{x^r}{r!}\right]$

For r = 0, 1, 2, 3, etc. is also discrete.

If n is large and q is small the Poisson Approximates the Binomial

Theorem (Equation 6):

If n is large and q is small, then $\frac{n!}{k!(n-k)!}p^{n-k}q^{k} \approx \frac{(nq)^{k}}{k!}e^{-nq}$

Proof :

$$\frac{n!}{k!(n-k)!}p^{n-k}q^{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}(1-q)^{n-k}q^{k} \quad \text{since } p=1-q$$

$$\approx \frac{n^{k}}{k!}(1-q)^{n-k}q^{k} \quad \text{since n is large}$$

$$= \frac{n^{k}}{k!}\cdot\frac{(1-q)^{n}}{(1-q)^{k}}\cdot q^{k} \approx \frac{n^{k}}{k!}\cdot\frac{(1-q)^{n}}{1}\cdot q^{k} = \frac{(nq)^{k}(1-q)^{n}}{k!} \quad \text{since q is small} \quad (1)$$

now compare $(1-q)^{n}$ with e^{-nq} by expanding each of them.

$$(1-q)^{n} = 1-nq + \frac{n(n-1)}{2!}q^{2} - \frac{n(n-1)(n-2)}{3!}q^{3} + \cdots$$

$$\approx 1-nq+\frac{n^{2}}{2!}q^{2}-\frac{n^{3}}{3!}q^{3}+\cdots = 1-nq+\frac{(nq)^{2}}{2!}-\frac{(nq)^{3}}{3!}+\cdots \quad \text{since n is large}$$

$$\therefore \ (1-q)^{n} \approx 1-nq+\frac{(nq)^{2}}{2!}-\frac{(nq)^{3}}{3!}+\cdots \quad (2)$$

$$e^{-nq} = 1-nq + \frac{(nq)^{2}}{2!} - \frac{(nq)^{3}}{3!}+\cdots \quad (3)$$

comparing (2) and (3) $\Rightarrow (1-q)^{n} \approx e^{-nq}$ (4)

Replacing e^{-nq} for $(1-q)^{n}$ in (1) we get $\frac{n!}{k!(n-k)!}p^{n-k}q^{k} \approx \frac{(nq)^{k}}{k!}e^{-nq}$ //

Proving Equation 6 using L'Hospital's Rule.
Theorem 1:

$$\lim_{x\to 0} 1-e^{-x} = x \quad \text{or} \quad \lim_{x\to 0} 1-x = e^{-x}$$

Proof

By using L'Hospital's Rule we see the relationship between x and $1-e^{-x}$ as x gets very small.

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \frac{1-e^{-x}}{x} = \frac{0}{0} \text{ and } \lim_{x\to 0}\frac{f'(x)}{g'(x)} = \frac{e^{-x}}{1} = 1 \Rightarrow \lim_{x\to 0}\frac{1-e^{-x}}{x} = 1 \Rightarrow$$

$$\lim_{x\to 0} 1-e^{-x} = x \quad \text{or} \quad \lim_{x\to 0} 1-x = e^{-x}$$

Theorem 2:

If n is very large, and q is very small, then

$$P_n(k) = \left(\frac{n!}{k!(n-k)!}\right)(1-q)^{n-k}q^k \xrightarrow[n\to\infty]{} \frac{(nq)^k}{k!}e^{-nq}$$

Proof

$$P_n(0) = (1-q)^n$$

$$P_n(1) = n(1-q)^{n-1}q$$

$$P_n(2) = \frac{n(n-1)(1-q)^{n-2}q^2}{2!}$$

$$P_n(3) = \frac{n(n-1)(n-2)(1-q)^{n-3}q^3}{3!}, \text{ etc. by definition.}$$

Since n is very large we can replace n–k with n and get

$$P_n(0) = (1-q)^n$$

$$P_n(1) = n(1-q)^n q$$

$$P_n(2) \cong \frac{n^2(1-q)^n q^2}{2!}$$

$$P_n(3) \cong \frac{n^3(1-q)^n q^3}{3!}, \text{ etc.}$$

Since q is very small we can replace (1–q) with e^{-q} by L'Hospital's Rule. Therefore

$$P_n(0) \cong (e^{-q})^n = e^{-nq}$$

$$P_n(1) \cong n(e^{-q})^n q = nqe^{-nq}$$

$$P_n(2) \cong \frac{n^2(e^{-q})^n q^2}{2!} = \frac{(nq)^2 e^{-nq}}{2!}$$

$$P_n(3) \cong \frac{n^3(e^{-q})^n q^3}{3!} = \frac{(nq)^3 e^{-nq}}{3!}, \text{ etc. //}$$