Chapter 8

Numerical Analysis

Introduction (Needs Work)

When large amounts of failure data of a particular device are available, obtaining an accurate failure distribution curve is a relatively straightforward process using standard statistical techniques. However, when failure data is scarce, various assumptions are made, and oftentimes curve fitting techniques are employed.

1. The primary advantage of Weibull analysis is the ability to provide reasonably accurate failure analysis and failure forecasts with extremely small samples.
2. Another advantage of Weibull analysis is that it provides a simple and useful graphical plot.

In addition to the above, Weibull has another advantage. Weibull equations are integrable, and can be designed to curve fit (approximate) numerical integrations of various non-integrable functions. One of which is the Normal failure density function (fdf). What this means is that a Normal Probability of Failure Function, which is an integration of the Normal fdf, can be expressed with the luxury of an equation as opposed to an integration process. This feature of Weibull becomes very attractive when numerical integration tools are not available. This feature is illustrated by the following problem:

Example 1 Weibull P_f Approximation of a Normal Distribution

A tire has a u = mean life of 100,000 miles, with s = sigma of 20,000 miles, and with k = miles already logged = 5000 miles. Derive the P_f equation as a function of t miles.

This approach uses a process of solving three simultaneous equations based on three known points of intersection of the actual Normal P_f curve. It is somewhat more complicated than the "1 point" method, but yields much more accurate results.

The Weibull equation is $P_f = F(t) = 1 - e^{-\left[\left(\frac{t-b}{a}\right)^c\right]}$ with $c = 3.4564775$, $a = 2ns \cdot s / m$, where $m = 1.00545149$,

ns is a real number (selected for best fit) such that $0 < ns < 3.9$, and $b = u - (a[-\ln(0.5)]^{\frac{1}{c}} + k)$.

Note the above P_f equation is not defined for $t-b < 0$. Therefore this method must be "piecewise" defined. Define $P_f = 0$ for $0 \le t < b$, and P_f as shown on the graph below for $b \le t$.

In this example $P_f = 1 - e^{-\left[\left(\frac{t-32.384}{69.620}\right)^{3.456}\right]}$ using $u = 100$, $s = 20$, $k = 5$, and $ns = 1.75$.

Note that the above P_f equation is not defined for $t < b = 32.384$. Therefore this method must be "piecewise" defined. Define $P_f = 0$ for $0 \le t < b$, and P_f as shown above for $b \le t$.

The following is a graph of the tire P_f calculated using a numerical integration technique together with the above Weibull approximation. Notice the very close fit.

Graph of Modified Weibull Approximation vs. Normal P_f
(mean = 100000, sigma = 20000, miles logged = 5000)

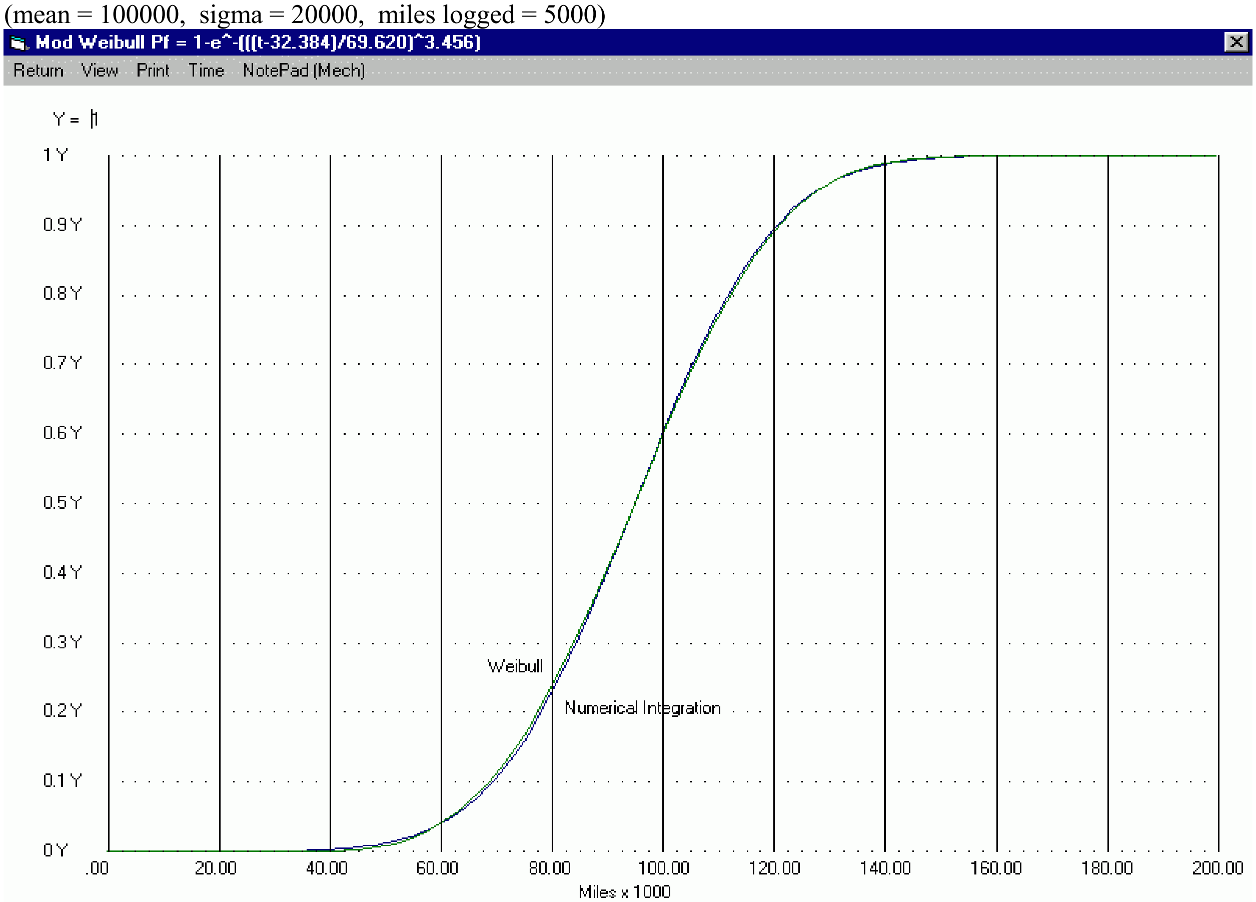


Figure 7-1

Derivation of the Weibull Method of Approximating a Normal Failure Distribution
This discussion describes the details of how the parameters of the Weibull Method were derived. It is quite lengthy and can be skipped over without loss of any information pertaining to its applications.

The approximation equation $F(t) = 1 - e^{-\left[\left(\frac{t-b}{a}\right)^{c}\right]}$, can be constructed to intersect the Normal P_f Curve at three predetermined points. The three points of intersection are designed to occur at $P_f = 0.5 - ar$, $P_f = 0.5$, and $P_f = 0.5 + ar$. The term ar (area under the Standard Normal Curve) corresponds to a selected ns (normalized sigma) which defines two of the three points. ns = 1 corresponds to an ar = .3413, ns = 2 corresponds to ar = .4772 etc. as listed in a Standard Normal Distribution Table. So for example if a and c were calculated using ns = 1.75, the Modified Weibull Curve will intersect the Normal P_f Curve at $P_f = (0.5 - 0.4599)$, $P_f = (0.5)$, $P_f = (0.5 + 0.4599)$ in other words ½ ± 1.75 sigma of area.
This approach involves the solution of three simultaneous equations with three unknowns.

Define $P_f = 1 - e^{-\left[\left(\frac{t-b}{a}\right)^{c}\right]}$ such that $P_f = 0.5$ when $t = u - k$,
$P_f = 0.5 + ar$ when $t = u - k + ns \cdot s$, and $P_f = 0.5 - ar$ when $t = u - k - ns \cdot s \Rightarrow$

$$0.5 = 1 - e^{-\left[\left(\frac{u-k-b}{a}\right)^{c}\right]} \Rightarrow e^{-\left[\left(\frac{u-k-b}{a}\right)^{c}\right]} = 0.5 \Rightarrow u - k - b = a\,[-\ln(0.5)]^{\frac{1}{c}} \qquad (1)$$

$$0.5 + ar = 1 - e^{-\left[\left(\frac{u-k+ns\cdot s-b}{a}\right)^{c}\right]} \Rightarrow e^{-\left[\left(\frac{u-k+ns\cdot s-b}{a}\right)^{c}\right]} = 0.5 - ar \qquad (2)$$

$$0.5 - ar = 1 - e^{-\left[\left(\frac{u-k-ns\cdot s-b}{a}\right)^{c}\right]} \Rightarrow e^{-\left[\left(\frac{u-k-ns\cdot s-b}{a}\right)^{c}\right]} = 0.5 + ar \qquad (3)$$

$$(2)\text{ and }(3) \Rightarrow u - k + ns \cdot s - b = a\,[-\ln(0.5 - ar)]^{\frac{1}{c}}, \quad u - k - ns \cdot s - b = a\,[-\ln(0.5 + ar)]^{\frac{1}{c}} \Rightarrow \qquad (4)$$

$$b = u - k + ns \cdot s - a\,[-\ln(0.5 + ar)]^{\frac{1}{c}} = u - k - ns \cdot s - a\,[-\ln(0.5 - ar)/a]^{\frac{1}{c}} \Rightarrow$$

$$2 \cdot ns \cdot s = a\,[-\ln(0.5 - ar)]^{\frac{1}{c}} - a\,[-\ln(0.5 + ar)]^{\frac{1}{c}} \Rightarrow 2 \cdot ns \cdot s / a = [-\ln(0.5 - ar)]^{\frac{1}{c}} - [-\ln(0.5 + ar)]^{\frac{1}{c}} \Rightarrow$$

$$(5) = (6) \Rightarrow m = 2q \Rightarrow [-\ln(0.5 - ar)]^{\frac{1}{c}} - [-\ln(0.5 + ar)]^{\frac{1}{c}} = 2\left([-\ln(0.5)]^{\frac{1}{c}} - [-\ln(0.5+ar)]^{\frac{1}{c}}\right) \Rightarrow$$

$$a = 2ns{\cdot}s / m \text{ where } m = [-\ln(0.5 - ar)]^{\frac{1}{c}} - [-\ln(0.5 + ar)]^{\frac{1}{c}} \qquad (5)$$

$\boxed{\text{Note: ns = 1.75 => a = 3.48078264044765s}}$

$$\text{from (4) and (1)} \quad u - k - ns \cdot s - b = a\,[-\ln(0.5 + ar)]^{\frac{1}{c}} \quad \text{and} \quad u - k - b = a\,[-\ln(0.5)]^{\frac{1}{c}} \Rightarrow$$

$$a = ns{\cdot}s / q \text{ where } q = [-\ln(0.5)]^{\frac{1}{c}} - [-\ln(0.5 + ar)]^{\frac{1}{c}} \qquad (6)$$

$$2\,[-\ln(0.5)]^{\frac{1}{c}} = [-\ln(0.5+ar)]^{\frac{1}{c}} + [-\ln(0.5-ar)]^{\frac{1}{c}} \qquad \text{(Note that c is a function of ar.)} \tag{7}$$

The equation (7) can be solved using a Taylor Series method which is a subject for another paper. Rather than a time consuming development, recall that c is a function of ar, ar is a function of ns (normalized sigma), and therefore c is a function of ns. This function was obtained utilizing the above equation in conjunction with a curve fitting method using the following data.

ns	ar	c
.5	0.191462461275692	3.27504943957515
1	0.34134474609543	3.32363473186359
1.5	0.43319279877211	3.4045530388795
1.75	0.459940843131247	3.45680410401036
2	0.477249867955804	3.51649846691278
2.5	0.493790334365092	3.65613580018552

$$\boxed{c = -0.006224132ns^3 + 0.089994945ns^2 - 0.031275449ns + 3.269690806 \quad \text{where } 0 < ns < 3.9} \tag{8}$$

Note: ns = 1.75 was chosen for the calculation of c to force intersections at ½ ± 1.75 sigma of area.
ns = 1.75 => c = 3.45721082957605
The following equation for b will force the Modified Weibull Curve to intersect at $P_f = 0.5$. Recall that b is solved such that $P_f = 0.5$ when t = u – k.

By substitution of $P_f = 0.5$, t = u – k in $P_f = 1 - e^{-\left[\left(\frac{t-b}{a}\right)^c\right]}$ we get $0.5 = 1 - e^{-\left[\left(\frac{u-k-b}{a}\right)^c\right]} \Rightarrow$

$e^{-\left[\left(\frac{u-k-b}{a}\right)^c\right]} = 0.5 \Rightarrow (u-k-b)/a = [-\ln(0.5)]^{\frac{1}{c}} \Rightarrow b = u - (a\,[-\ln(0.5)]^{\frac{1}{c}} + k) \Rightarrow$

$\boxed{b = u - 3.130874117s - k \text{ for } ns = 1.75}$

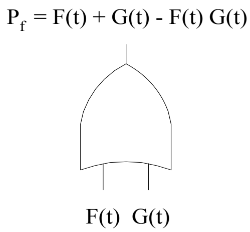
Therefore given a mechanical device whose Normal Distribution of failure has a mean = u, and a sigma = s, and k = the number of units (miles, hours, etc) already logged on the device, then using ns = 1.75, a = 3.48078264s, b = u – 3.130874117s – k, c = 3.457210829, and

$P_f = 1 - e^{-\left[\left(\frac{t - u + 3.130874117s + k}{3.48078264\,s}\right)^{3.457210829}\right]}$ for $t \ge u - 3.130874117s$ and

$P_f = 0$ for $t < u - 3.130874117s$

Example 2 Weibull P_f Approximation of Two Mechanical Devices in Series
A motorcycle has tire1 with u=100, sigma = 20, and 10k miles logged. The other tire2 has a u=90, sigma = 30, and 20k miles logged. Calculate the P_f due to wearout using the Modified Weibull Approximation Method.

Logic Diagram



F(t) = tire1P_f and G(t) = tire2 P_f

Selecting ns = 1.75 => a = 3.48078264044765s, b = u – 3.130874117s – k, and c = 3.4564775,

$$F(t) = 1 - e^{-\left[\left(\frac{t-b1}{a1}\right)^{c}\right]} \quad a1 = 69.620464 \text{ and } b1 = 27.383926$$

$$G(t) = 1 - e^{-\left[\left(\frac{t-b2}{a2}\right)^{c}\right]} \quad a2 = 104.430697 \text{ and } b2 = -23.924111$$

$$P_f = F(t) + G(t) - F(t)G(t) = 1 - e^{-\left[\left(\frac{t-b1}{a1}\right)^{c} + \left(\frac{t-b2}{a2}\right)^{c}\right]}$$

$$\Rightarrow P_f = 1 - e^{-\left[\left(\frac{t-27.384}{69.620}\right)^{3.456} + \left(\frac{t+23.924}{104.431}\right)^{3.456}\right]}$$

Note that the above equation is not defined for t < 27.384 and therefore the P_f must be "piecewise" defined as follows:

$$P_f = 1 - e^{-\left[\left(\frac{t+23.924}{104.430}\right)^{3.456}\right]} \quad \text{for } t < 27.384$$

$$\text{and } P_f = 1 - e^{-\left[\left(\frac{t-27.384}{69.620}\right)^{3.456} + \left(\frac{t+23.924}{104.431}\right)^{3.456}\right]} \quad \text{for } 27.384 \le t$$

See Figure 7-2

Modified Weibull Approx. vs. NI of 2 Mechanical (Normal) Devices in Series

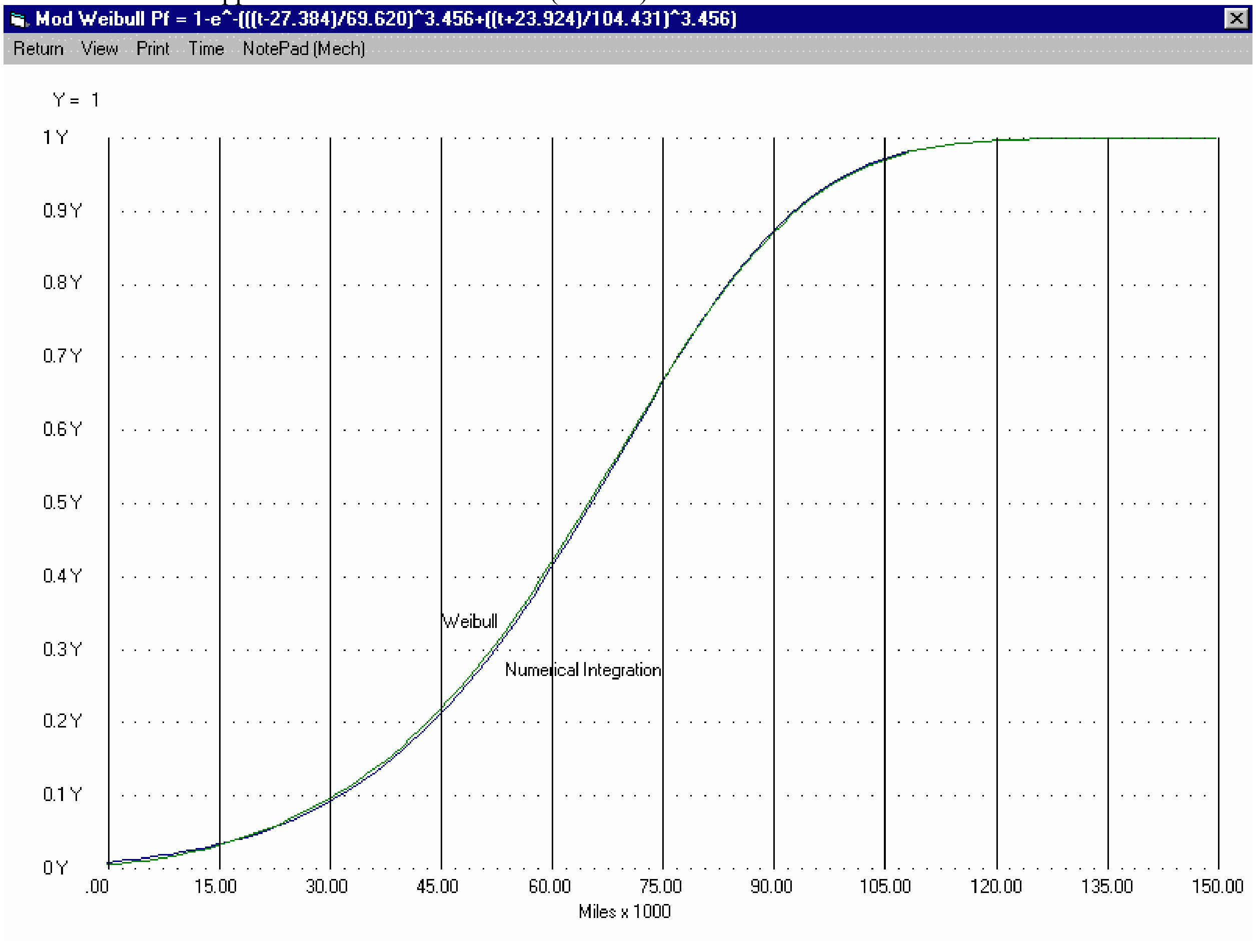
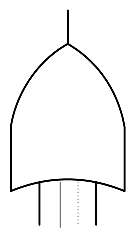


Figure 7-2

Example 3 Weibull P_f Approximation of n Mechanical Devices in Series

A certain system (machine) is made up of n mechanical devices. System failure is defined to be a failure of any one of the n devices. Device 1 has a mean of u_1, a sigma of s_1, and hours logged = k_1. Similarly device 2 has a u_2, s_2, and a k_2, and device n has a u_n, s_n, and a k_n. Derive an equation to calculate the P_f due to wearout of this system.

Logic Diagram



Where $f_1(t)$ = device1 P_f , $f_2(t)$ = device 2 P_f and $f_n(t)$ = device n P_f $\Rightarrow$

$$f_1(t) = 1 - e^{-\left[\left(\frac{t-b_1}{a_1}\right)^c\right]} \text{ where } a_1 = 3.48078264044765 s_1 \text{ and } b_1 = u_1 - k_1 - a_1[-\ln(0.5)]^{\frac{1}{c}} \text{ and}$$

$$f_2(t) = 1 - e^{-\left[\left(\frac{t-b_2}{a_2}\right)^c\right]} \text{ where } a_2 = 3.48078264044765 s_2, \text{ and } b_2 = u_2 - k_2 - a_2[-\ln(0.5)]^{\frac{1}{c}}$$

$$f_n(t) = 1 - e^{-\left[\left(\frac{t-b_n}{a_n}\right)^c\right]} \text{ where } a_n = 2ns \cdot s_n / m, \text{ and } b_n = u_n - k_n - a_n[-\ln(0.5)]^{\frac{1}{c}}$$

$$r_i(x) = \frac{c(x-b_i)^{c-1}}{a_i{}^c}, \quad \int_0^t r_i(x)dx = \left(\frac{t-b_i}{a_i}\right)^c, \quad \text{and } P_f(sys) = 1 - e^{-\left(\sum_{i=1}^{n}\int_0^t r_i(x)dx\right)} \quad \Rightarrow$$

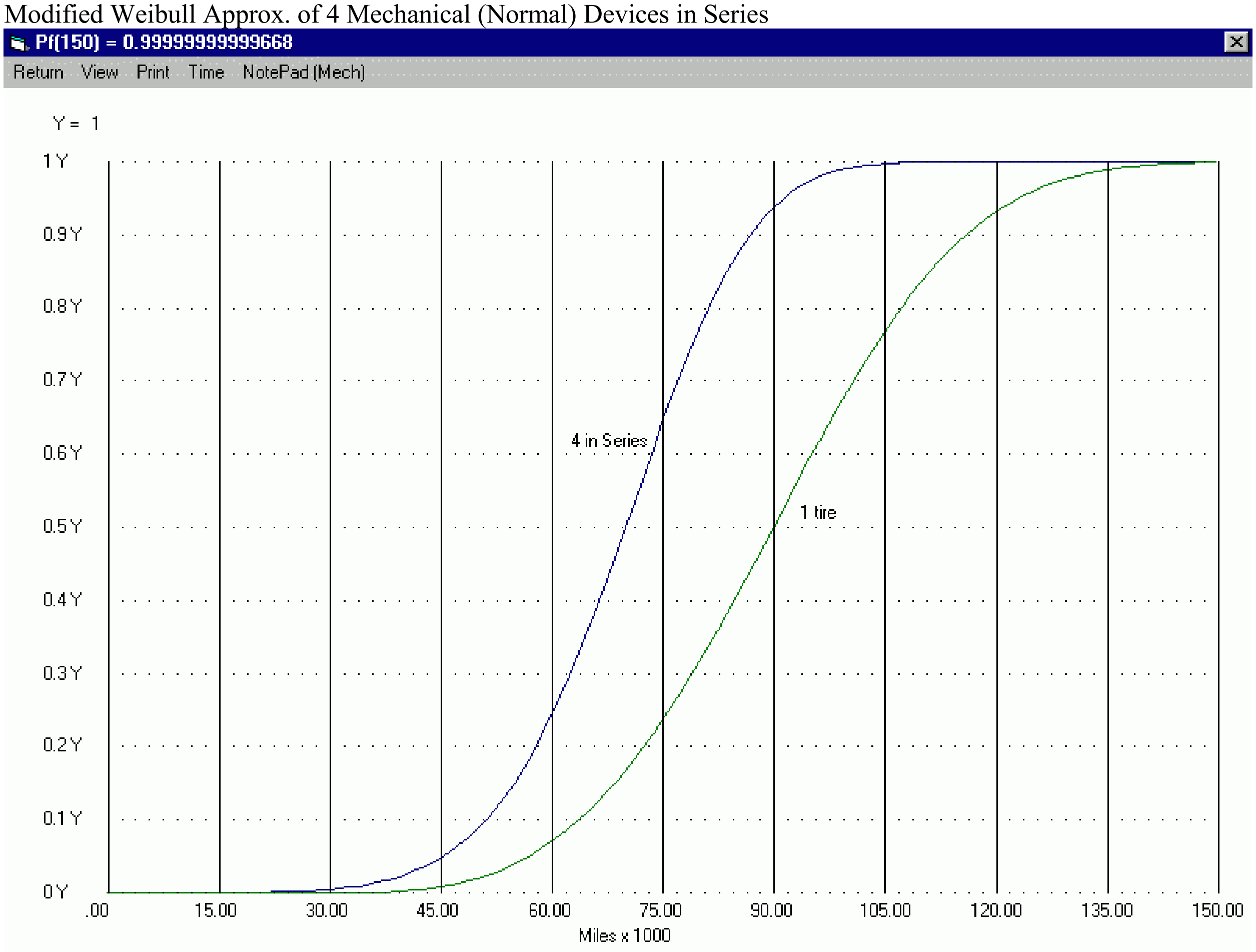
$$P_f(sys) = 1 - e^{-\left[\left(\frac{t-b_1}{a_1}\right)^c + \left(\frac{t-b_2}{a_2}\right)^c + \cdots + \left(\frac{t-b_n}{a_n}\right)^c\right]}$$

Example 4 Four Mechanical Devices in Series
A car has 4 tires. Assume for the sake of simplicity all 4 tires have a u = 100, sigma = 20, and 10k miles logged. Calculate the P_f due to wearout.

$$P_f = 1 - e^{-\left[\left(\frac{t-b}{a}\right)^c + \left(\frac{t-b}{a}\right)^c + \left(\frac{t-b}{a}\right)^c + \left(\frac{t-b}{a}\right)^c\right]} = 1 - e^{-\left[4\left(\frac{t-b}{a}\right)^c\right]}$$

where $c = 3.456,\ a = 2ns \cdot s / m,\ b = u - k - a\,[-\ln(0.5)]^{\frac{1}{c}} \Rightarrow P_f = 1 - e^{-\left[4\left(\frac{t-27.384}{69.620}\right)^{3.456}\right]}$
where t is measured in units of 1000 miles.

Modified Weibull Approx. of 4 Mechanical (Normal) Devices in Series

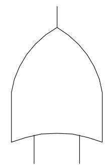


Example 5 Mechanical Device in Series with an Electrical Device

A simple system is made up of a mechanical device in series with an electrical device. The mechanical device has u = 100, sigma = 20, with 10 hours logged. The electrical device has a failure rate of .01 failures per hour. Calculate the P_f of the system.

Logic Diagram

$P_f = F(t) + G(t) - F(t)\,G(t)$



F(t) G(t)

where $F(t) = P_f\,(\text{mech})$ and $G(t) = P_f\,(\text{elect.})$ =>

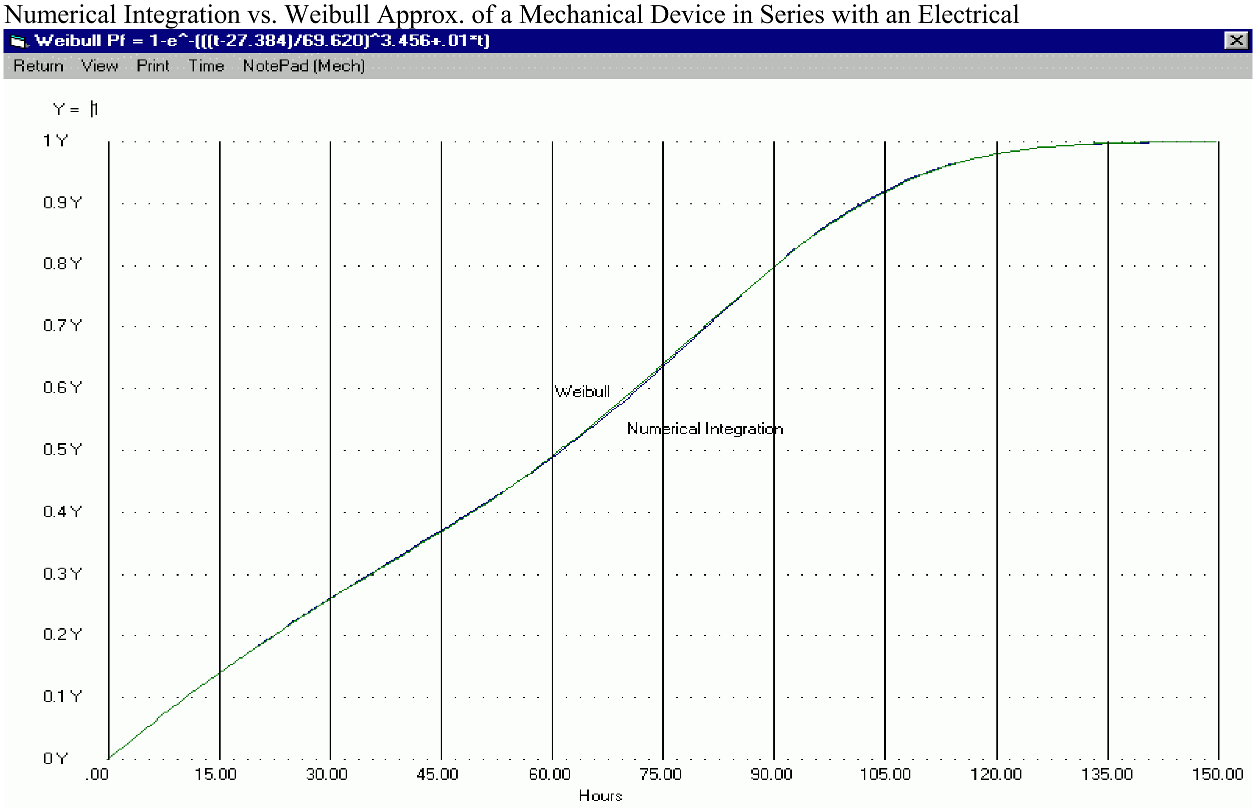
$$F(t) = 1 - e^{-\left[\left(\frac{t-b}{a}\right)^{c}\right]} \quad \text{where } a = 2ns \cdot s / m, \text{ and } b = u - k - a\,[-\ln(0.5)]^{\frac{1}{c}} \text{ and}$$

$$G(t) = 1 - e^{-\lambda t} \quad \text{where } \lambda = .01$$

$$\text{Now } P_f = F(t) + G(t) - F(t)G(t) = 1 - e^{-\left[\left(\frac{t-b}{a}\right)^{c} + \lambda t\right]} \text{ and}$$

$$u = 100,\ s = 20,\ k = 10,\ \lambda = .01 \ \Rightarrow\ P_f = 1 - e^{-\left[\left(\frac{t-27.384}{69.620}\right)^{3.456} + .01t\right]}$$

Numerical Integration vs. Weibull Approx. of a Mechanical Device in Series with an Electrical



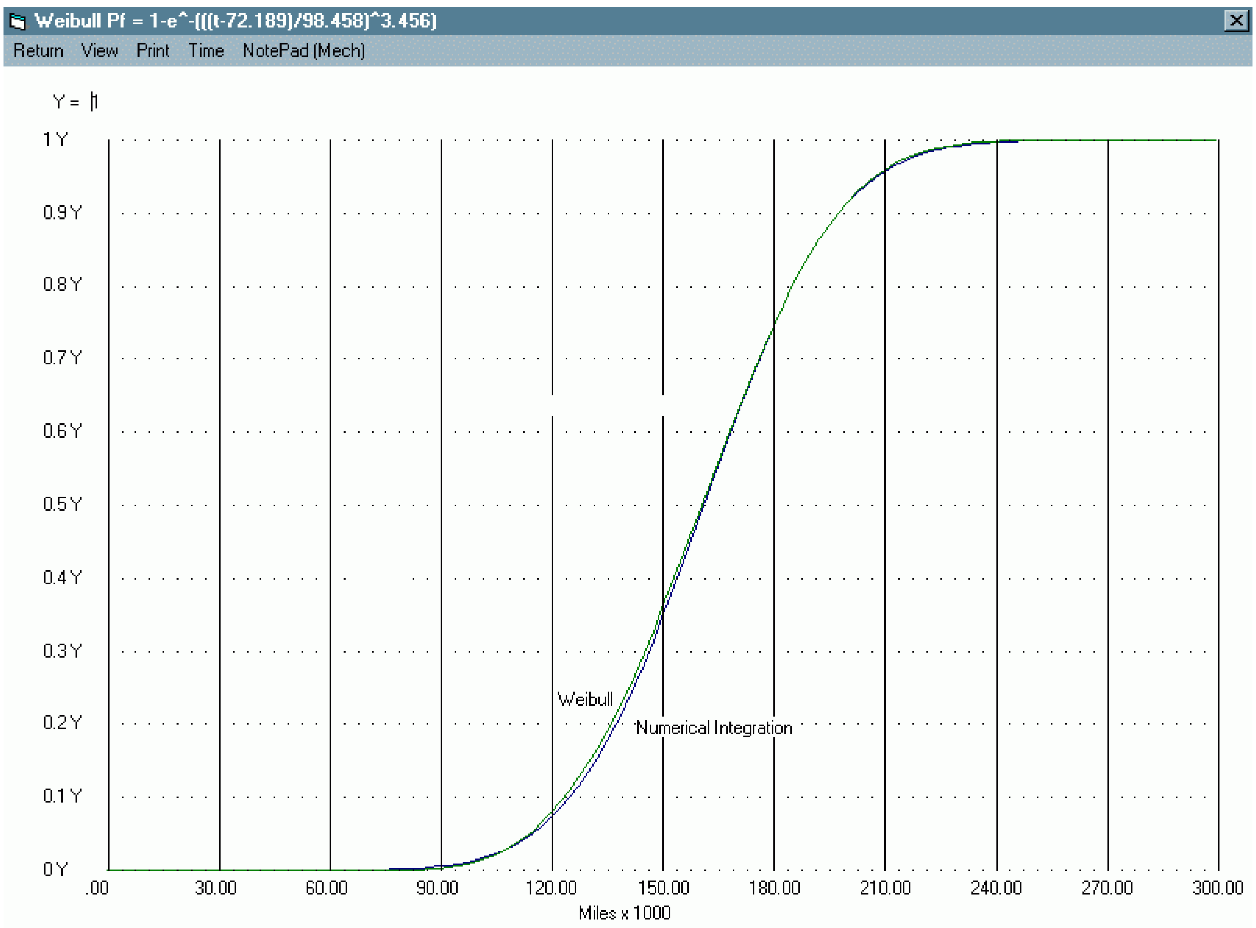
Example 6 Mechanical Device in Standby with a Mechanical Device
A simple system is made up of one mechanical device in operation with a second mechanical device on standby. Upon detection of failure of the operating device, the second device is immediately turned on and placed into operation. The first mechanical device has u = 100, sigma = 20, with 10 hours logged. The standby mechanical device has u = 100, sigma = 20, with 30 hours previously logged. Calculate the P_f of the system.

Set $u_1 = 100$, $s_1 = 20$, $k_1 = 10$, $u_2 = 100$, $s_2 = 20$, and $k_2 = 30$, ns = 1.75, m = 1.00545148831.

$$P_f\,(\text{standby}) = 1 - e^{-\left[\left(\frac{t-b}{a}\right)^c\right]} \quad \text{where } c = 3.4564775, \quad a = \frac{2ns}{m}\sqrt{s_1{}^2 + s_2{}^2} \quad \text{and}$$

$$b = u_1 + u_2 - (k_1 + k_2) - a\,[-\ln(0.5)]^{\frac{1}{c}} \quad \Rightarrow \quad P_f\,(\text{standby}) = 1 - e^{-\left[\left(\frac{t-71.447}{98.458}\right)^{3.456}\right]}$$

To be redone



Numerical Integration vs. Weibull Approx.
Mechanical Device in Standby with a Mechanical Device

Figure 7-

Example 7 n Mechanical Devices in Series with 1 on Standby
This example outlines a procedure for solving the classic problem as outlined here:
A car has 4 tires with one spare in the trunk. One tire goes flat (fails). The flat is removed and the spare tire is installed. The car travels on until another flat occurs. Calculate the probability of the entire event.

Let $F(t) = 1 - e^{-\left[\left(\frac{t-b_1}{a_1}\right)^c\right]} = P_f\,(\text{first flat}) = P_f\,(\text{4 tires in series})$. Let $G(t) = 1 - e^{-\left[\left(\frac{t-b_2}{a_2}\right)^c\right]} = P_f\,(\text{spare})$

From Convolution Theorem $P_f\,(\text{sys}) = \int_0^t F'(x)\,G(t-x)dx$ where $F'(x) = \dfrac{c(x-b_1)^{c-1}}{{a_1}^c}\; e^{-\left[\left(\frac{t-b_1}{a_1}\right)^c\right]} \Rightarrow$

$$P_f\,(\text{sys}) = \frac{c}{{a_1}^c}\int_0^t (x-b_1)^{c-1}\; e^{-\left[\left(\frac{t-b_1}{a_1}\right)^c\right]'}\,(1-e^{-\left[\left(\frac{t-b_2}{a_2}\right)^c\right]})\,dx$$

Derivation of a Mechanical Device in Standby with a Mechanical Device

Given a simple system of two normally distributed devices with parameters u1, s1, k1, and u2, s2, k2, with one device in standby of the other, then the system probability failure distribution is normal with u, s, and k, such that $u = u_1 + u_2$, $s = \sqrt{s_1^2 + s_2^2}$, and $k = k_1 + k_2$.

Recall from Example 1 a = 2ns·s / m, and $b = u - (a\,[-\ln(0.5)]^{\frac{1}{c}} + k)$. Then by substitution

a = 2ns·s / m $\Rightarrow$ $a = \frac{2ns}{m}\sqrt{s_1^2 + s_2^2}$ and

$b = u - (a\,[-\ln(0.5)]^{\frac{1}{c}} + k) \;\Rightarrow\; b = u_1 + u_2 - (a\,[-\ln(0.5)]^{\frac{1}{c}} + k_1 + k_2)$

Derivation of the Weibull Method of Approximating an Electrical Standby Electrical
This discussion describes the mathematical details of how the parameters of the Weibull Method were derived. It is quite lengthy and can be skipped over without loss of any information pertaining to its applications.

The approximation equation $F(t) = 1 - e^{-\left[\left(\frac{t-b}{a}\right)^{c}\right]}$, can be constructed to intersect the Standby P_f Curve at three predetermined points. The three points of intersection are designed to occur at t_1, t_2, and t_3. This approach involves the solution of three simultaneous equations with three unknowns.

Given an electrical device in operation with failure rate λ, with another electrical device powered off on standby with failure rate β, calculate the probability of a system failure.

$$P_f(\text{standby}) = 1 - \frac{\beta}{\beta-\lambda}e^{-\lambda t} + \frac{\lambda}{\beta-\lambda}e^{-\beta t} = 1 - e^{-\left[\left(\frac{t-b}{a}\right)^{c}\right]} \Rightarrow$$

$$e^{-\left[\left(\frac{t-b}{a}\right)^{c}\right]} = \frac{\beta}{\beta-\lambda}e^{-\lambda t} - \frac{\lambda}{\beta-\lambda}e^{-\beta t} = f(t) \Rightarrow e^{-\left[\left(\frac{t_i-b}{a}\right)^{c}\right]} = f(t_i) \text{ for i = 1, 2, 3 =>}$$

$$b = t_i - a(-\ln[f(t_i)])^{\frac{1}{c}} \Rightarrow (t_2 - t_1)/a = (-\ln[f(t_2)])^{\frac{1}{c}} - (-\ln[f(t_1)])^{\frac{1}{c}} \text{ and} \qquad (1)$$

$$(t_3 - t_2)/a = (-\ln[f(t_3)])^{\frac{1}{c}} - (-\ln[f(t_2)])^{\frac{1}{c}}$$ Selecting t_1,t_2, and t_3 such that $t_2 - t_1 = t_3 - t_2$

$$(-\ln[f(t_2)])^{\frac{1}{c}} - (-\ln[f(t_1)])^{\frac{1}{c}} = (-\ln[f(t_3)])^{\frac{1}{c}} - (-\ln[f(t_2)])^{\frac{1}{c}} \Rightarrow$$

From (1) $$a = \frac{t_2 - t_1}{(-\ln[f(t_2)])^{\frac{1}{c}} - (-\ln[f(t_1)])^{\frac{1}{c}}}$$

Again from (1) $$b = t_2 + \frac{(t_1 - t_2)(-\ln[f(t_2)])^{\frac{1}{c}}}{(-\ln[f(t_2)])^{\frac{1}{c}} - (-\ln[f(t_1)])^{\frac{1}{c}}}$$

$$2(-\ln[f(t_2)])^{\frac{1}{c}} = (-\ln[f(t_3)])^{\frac{1}{c}} + (-\ln[f(t_1)])^{\frac{1}{c}}$$ See Appendix for calculation of c.

Derivation of the Weibull Method of Approximating a Lognormal Failure Distribution

This discussion describes the mathematical details of how the parameters of the Weibull Method for the Lognormal Failure Distribution were derived. Recall that the Lognormal Distribution is a Normal Distribution of ln(t) as opposed to t. Therefore substitute ln(t) for t in the Weibull F(t) equation and get

$F(t) = 1 - e^{-\left[\left(\frac{\ln(t)-b}{a}\right)^{c}\right]}$. However this equation would be correct if k=0 i.e. if the device had no previous hours (miles) logged. To make the adjustment

define $P_f = 1 - e^{-\left[\left(\frac{\ln(t+k)-b}{a}\right)^{c}\right]}$ such that a and c are the same as for the Normal calculatio n, and

$b = u - a[-\ln(0.5)]^{\frac{1}{c}}$

Derivation of the Polynomial Method of Approximating a Normal Failure Distribution

Given a Normal Failure Distribution with mean u and sigma s, then the P_f can be very closely approximated by

$$P_f = F(t) = 1 - e^{\frac{1}{2}[P(t) - \mathrm{abs}(P(t))]} \quad \text{where } P(t) = a(t+hl)^3 + b(t+hl)^2 + c(t+hl) + d,\ \ hl = \text{hours (units) previously logged,}$$

$$\begin{aligned}
a &= -\,0.40861530764759/s^3 \\
b &= 0.122584592294277\ u/s^3 - 0.305630571871765/s^2 \\
c &= -\,0.122584592294277\ u^2/s^3 + 0.61126114374353\ u/s^2 - 0.782437098467209/s \\
d &= 0.0408615307647492\ u^3/s^3 - 0.30560571871749\ u^2/s^2 + 0.782437098467209\ u/s - 0.693147180559944
\end{aligned} \tag{1}$$

Coefficients a, b, c, and d were derived as follows:
Assume $P_f = 1 - e^{P(t)}$ where P(t) is a polynomial $\Rightarrow Ps = e^{P(t)} \Rightarrow P(t) = \ln(Ps)$. Four points on the t axis are strategically chosen such that $t_1 = u - 11s/4$, $t_2 = u - 7s/4$, $t_3 = u$, and $t_4 = u + 7s/4$. This yields four simultaneous equations $P(t_1) = \ln(Ps_1)$, $P(t_2) = \ln(Ps_2)$, $P(t_3) = \ln(Ps_3)$, $P(t_4) = \ln(Ps_4)$ where Ps_i = probability of success measured at ti. Note that Ps_i are constant for any u and s. In fact $Ps_1 = 0.997020236402405$, $Ps_2 = 0.959940843131253$, $Ps_3 = 0.5$, and $Ps_4 = 0.0400591568687467$ extracted from a Normal Cumulative Distribution Table. Therefore

$$\begin{aligned}
P(t_1) &= \ln(0.997020236402405) = -0.00298421193199815 \\
P(t_2) &= \ln(0.959940843131253) = -0.0408836181572305 \\
P(t_3) &= \ln(0.5) = -0.693147180559945 \\
P(t_4) &= \ln(0.0400591568687467) = -3.21739799567722
\end{aligned}$$

The strategy is to "curve fit" the four points $(t_i, \ln(Ps_i))$ with a polynomial curve of order three. The coefficients a, b, c, and d, of the polynomial $P(t) = at^3 + bt^2 + ct + d$ can be derived using a curve fitting software program. For example if u = 100 and s = 20, the result will be

$$P(t) = -0.0000051076913456t^3 + 0.000768230973998t^2 - 0.0395373093552962t + 0.685965360578848$$

However, the software program will not derive a, b, c, and d in algebraic form as shown above. Every time u or s is changed, the program would have to be rerun to determine the new P(t). The derivation of the coefficients in algebraic form can be accomplished utilizing the method for solving a set of simultaneous equations using determinants.

$$\text{Let}\ |A| = \begin{vmatrix} t_1^3 & t_1^2 & t_1 & 1 \\ t_2^3 & t_2^2 & t_2 & 1 \\ t_3^3 & t_1^2 & t_1 & 1 \\ t_4^3 & t_4^2 & t_4 & 1 \end{vmatrix} = \begin{vmatrix} (u-11s/4)^3 & (u-11s/4)^2 & (u-11s/4) & 1 \\ (u-7s/4)^3 & (u-7s/4)^2 & (u-7s/4) & 1 \\ u^3 & u^2 & u & 1 \\ (u+7s/4)^3 & (u+7s/4)^2 & (u+7s/4) & 1 \end{vmatrix} = \frac{33957}{256}s^6$$

$$\text{Let}\ |A_1| = \begin{vmatrix} P(t_1) & (u-11s/4)^2 & (u-11s/4) & 1 \\ P(t_2) & (u-7s/4)^2 & (u-7s/4) & 1 \\ P(t_3) & u^2 & u & 1 \\ P(t_4) & (u+7s/4)^2 & (u+7s/4) & 1 \end{vmatrix} = -5.42005859444891s^3$$

$$|A_2| = \begin{vmatrix} (u-11s/4)^3 & P(t_1) & (u-11s/4) & 1 \\ (u-7s/4)^3 & P(t_2) & (u-7s/4) & 1 \\ u^3 & P(t_3) & u & 1 \\ (u+7s/4)^3 & P(t_4) & (u+7s/4) & 1 \end{vmatrix} = 16.2601757833467us^3 - 40.5402239415997s^4$$

$$|A_3| = \begin{vmatrix} (u-11s/4)^3 & (u-11s/4)^2 & P(t_1) & 1 \\ (u-7s/4)^3 & (u-7s/4)^2 & P(t_2) & 1 \\ u^3 & u^2 & P(t_3) & 1 \\ (u+7s/4)^3 & (u+7s/4)^2 & P(t_4) & 1 \end{vmatrix} = -16.2601757833468u^2s^3 + 81.0804478831994us^4 - 103.786002158793s^5$$

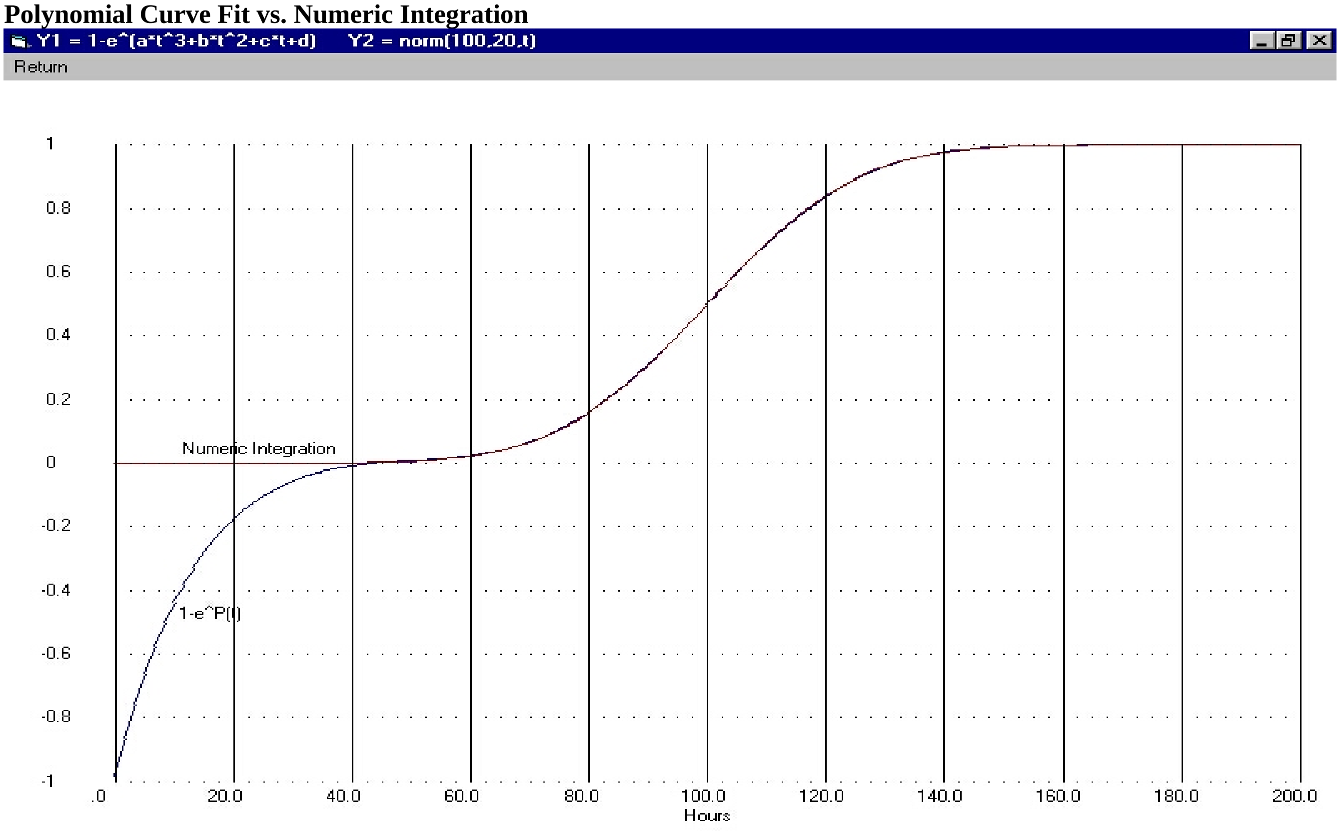
$$|A_4| = \begin{vmatrix} (u-11s/4)^3 & (u-11s/4)^2 & (u-11s/4) & P(t_1) \\ (u-7s/4)^3 & (u-7s/4)^2 & (u-7s/4) & P(t2) \\ u^3 & u^2 & u & P(t_3) \\ (u+7s/4)^3 & (u+7s/4)^2 & (u+7s/4) & P(t_4) \end{vmatrix} =$$

$$5.42005859444761\,u^3s^3 - 40.5402239415976\,u^2s^4 + 103.786002158792\,us^5 - 91.9421828526329\,s^6$$

and therefore $a = \dfrac{|A_1|}{|A|}$, $b = \dfrac{|A_2|}{|A|}$, $c = \dfrac{|A_3|}{|A|}$, $d = \dfrac{|A_4|}{|A|}$ as defined in (1).

The graph below compares a polynomial fit labeled “Y1 =1-e^P(t)” with a numeric integration of a Cumulative Normal Distribution Curve with u = 100 and s = 20. Note the perfect fit for t > 42 along with the complete miss of the mark for t < 42. The fix is rather simple. Define the P(t) equal to zero whenever it is positive i.e. define
$Y1 = P_f =$

Polynomial Curve Fit vs. Numeric Integration



Numerical Analysis a Normal Failure Distribution (CDF)

Given a Normal Failure Distribution with mean u and sigma s, then the P_f can be very closely approximated by

$$\text{PDF of Normal Distribution} = f(x) = \frac{1}{\sqrt{2\pi}\,s} e^{-\left[\frac{1}{2}\left(\frac{x-u}{s}\right)^2\right]} \Rightarrow \text{CDF} = F(t) = \int_0^t \frac{1}{\sqrt{2\pi}\,s} e^{-\left[\frac{1}{2}\left(\frac{x-u}{s}\right)^2\right]} dx$$

$$\text{Let } g(x) = e^{-\left[\frac{1}{2}\left(\frac{x-u}{s}\right)^2\right]} \text{ and } z = -\frac{1}{2}\left(\frac{x-u}{s}\right)^2 \text{ then}$$

$$g(x) = e^z = 1 + z + z^2/2! + z^3/3! + z^4/4! + \cdots \Rightarrow$$

$$\int_0^t g(x)dx = \int_0^t dx - \frac{1}{2}\int_0^t \left(\frac{x-u}{s}\right)^2 + \frac{1}{2^2\cdot 2!}\int_0^t \left(\frac{x-u}{s}\right)^4 - \frac{1}{2^3\cdot 3!}\int_0^t \left(\frac{x-u}{s}\right)^6 + \frac{1}{2^4\cdot 4!}\int_0^t \left(\frac{x-u}{s}\right)^8 - \cdots$$

$$\text{Let } v = \frac{x-u}{s} \Rightarrow dx = s\,dv \text{ and } \int_0^t \left(\frac{x-u}{s}\right)^n dx = \int_{-u/s}^{(t-u)/s} s v^n dv = \frac{s}{n+1}\Big[v^{n+1}\Big]_{-u/s}^{(t-u)/s} = \frac{(t-u)^{n+1} + u^{n+1}}{(n+1)\cdot s^n} \Rightarrow$$

$$\int_0^t g(x)dx = t - \frac{(t-u)^3 + u^3}{3\cdot 2\cdot s^2} + \frac{(t-u)^5 + u^5}{5\cdot 2^2\cdot 2!\cdot s^4} - \frac{(t-u)^7 + u^7}{7\cdot 2^3\cdot 3!\cdot s^6} + \cdots \Rightarrow$$

$$\int_0^t g(x)dx = \sum_{n=0}^{\infty} (-1)^n \frac{(t-u)^{2n+1} + u^{2n+1}}{(2n+1)\cdot 2^n\cdot n!\cdot s^{2n}} \Rightarrow$$

$$F(t) = \int_0^t f(x)dx = \int_0^t \frac{1}{\sqrt{2\pi}\,s} g(x)dx = \frac{1}{\sqrt{2\pi}\,s}\int_0^t g(x)dx = \frac{1}{\sqrt{2\pi}\,s}\sum_{n=0}^{\infty}(-1)^n \frac{(t-u)^{2n+1} + u^{2n+1}}{(2n+1)\cdot 2^n\cdot n!\cdot s^{2n}} \Rightarrow$$

$$F(t) = \frac{t}{\sqrt{2\pi}\,s} + \frac{1}{\sqrt{2\pi}\,s}\sum_{n=1}^{\infty}(-1)^n \frac{(t-u)^{2n+1} + u^{2n+1}}{(2n+1)\cdot 2^n\cdot n!\cdot s^{2n}} \quad \text{Now } F(t) = 0.5 \text{ when } t = u \text{ and } u > 6s \Rightarrow$$

$$0.5 = \frac{u}{\sqrt{2\pi}\,s} + \frac{1}{\sqrt{2\pi}\,s}\sum_{n=1}^{\infty}(-1)^n \frac{u^{2n+1}}{(2n+1)\cdot 2^n\cdot n!\cdot s^{2n}} \Rightarrow \frac{1}{\sqrt{2\pi}\,s}\sum_{n=1}^{\infty}(-1)^n \frac{u^{2n+1}}{(2n+1)\cdot 2^n\cdot n!\cdot s^{2n}} = 0.5 - \frac{u}{\sqrt{2\pi}\,s}$$

$$\text{Now } F(t) = \frac{t}{\sqrt{2\pi}\,s} + \frac{1}{\sqrt{2\pi}\,s}\sum_{n=1}^{\infty}(-1)^n \frac{(t-u)^{2n+1}}{(2n+1)\cdot 2^n\cdot n!\cdot s^{2n}} + \frac{1}{\sqrt{2\pi}\,s}\sum_{n=1}^{\infty}(-1)^n \frac{u^{2n+1}}{(2n+1)\cdot 2^n\cdot n!\cdot s^{2n}}$$

$$= \frac{t}{\sqrt{2\pi}\,s} + \frac{1}{\sqrt{2\pi}\,s}\sum_{n=1}^{\infty}(-1)^n \frac{(t-u)^{2n+1}}{(2n+1)\cdot 2^n\cdot n!\cdot s^{2n}} + 0.5 - \frac{u}{\sqrt{2\pi}\,s}$$

$$= 0.5 + \frac{t-u}{\sqrt{2\pi}\,s} + \frac{1}{\sqrt{2\pi}\,s}\sum_{n=1}^{\infty}(-1)^n \frac{(t-u)^{2n+1}}{(2n+1)\cdot 2^n\cdot n!\cdot s^{2n}} \Rightarrow$$

$$\boxed{F(t) = 0.5 + \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)\cdot 2^n\cdot n!}\left(\frac{t-u}{s}\right)^{2n+1}} \quad \text{Note : Series converges when } \left|\frac{t-u}{s}\right| < 6$$

Define F(t) =1 for t > u + 6s and F(t) = 0 for t < u – 6s