Chapter 5

Markov Analysis
Utilizing State Sequencing

Combinatorial vs. Non-Combinatorial
Recall from Basic Probability the definition of an "r-combination". Given n objects, an r-combination is any selection of r out of n objects without regard to sequence or order of arrangement. The total number of r-combinations selected from n objects is notated as ${}_nC_r$.

Example 1 (Combinatorial Problem)
Given 3 objects A, B, and C. How many ways can 2 objects be selected from this set of 3. There are 3 ways: AB, AC, and BC. Note that there is no regard to ordering because the act selecting A then B is considered the same as selecting B then A. The number of ways of selecting 2 out of 3 objects is notated as ${}_3C_2 = 3$.

Example 2 (Combinatorial Problem)
A system is made up of three black boxes A, B, and C. All three boxes are powered up at the same time. System failure is defined to be the state of which any one box is failed. How many ways can the system maintain operation?
There is 1 way this system can operate i.e. $1 = {}_3C_1$ accounting for A, B, and C all operating.

Example 3 (Combinatorial Problem)
A system is made up of three black boxes A, B, and C. All three boxes are powered up at the same time. System failure is defined to be the state of which any two or more boxes have failed. a. How many ways can the system maintain operation?
There are 4 ways this system can operate. They are $1 = {}_3C_3$ accounting for A, B, and C operating, and $3 = {}_3C_2$ accounting for A and B, A and C, or B and C operating.
b. How many ways can the system fail?
There are 4 ways this system can fail. They are $3 = {}_3C_1$ accounting for A and B, A and C, or B and C failing, and $1 = {}_3C_0$ accounting for A, B, and C failing.

Example 4 (A non-combinatorial Problem)
A system is made up of two black boxes A and B that are powered up at the same time. System failure is defined to be the state of which both boxes are failed and A fails before B. Note: B failing before A is not considered as a system failure. How many ways can A fail before B?
There are only two ways both boxes can fail, A before B, or B before A. Therefore the answer to the above is one of the two ways.
Note: Example 3 is a permutation problem as opposed to a combination problem. The important thing to keep in mind is that this problem involves a specified sequence, and therefore classified as non-combinatorial.

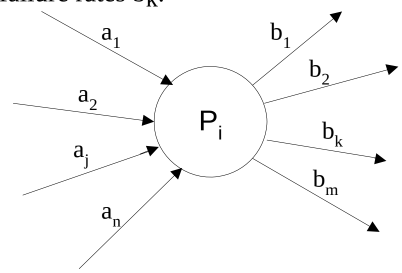
Example 5 (Non-combinatorial Problem)
A system is made up of two black boxes A and B. A is powered on, while B is powered off in standby mode. The instant A fails, box B is powered on replacing A. System failure is defined to be the state in which both A and B are failed. At any given time t, what is the probability that the system will be found operational?

Example 6 (Non-combinatorial Problem)
A system is made up of two identical resistors in parallel. System failure is defined to be the state in which both resistors are failed. The failure rate of both resistors are equal during normal operation. However when one resistor fails, the remaining resistor experiences a change in failure rate (increase) due to an increased loading effect. At any given time t, what is the probability that the system will be found operational?

Example 7 (Non-combinatorial Problem)
A system is made up of one black box with a known failure rate. Each time a failure is detected, the black box is repaired and put back in operation. The average time to repair is known. At any given time, what is the probability that the box will be found in operation?

Typical Markov State Diagram
A Markov diagram is a tool used to document (capture) the qualitative logic of a system's behavior. The following diagram shows a typical Markov State having n input transitions with constant failure rates a_j, and m output transitions with constant failure rates b_k.

transitions from other states



transitions to other states

There are several methods used for quantitative analysis, i.e. calculating Pi, the probability of being in state i. The best known method requires solutions to a set of simultaneous differential equations (DEs). The DEs are by-products of the State Diagram. Once the State Diagram is constructed i.e. once the qualitative analysis is performed on the system, the DEs manifest themselves, one for each state. The DE equation (general form) associated with each State i is written as follows:

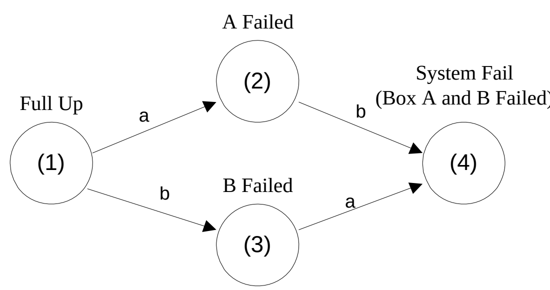
$$\frac{dP_i}{dt} = \sum_{j=1}^{n} a_j P_j - \left(\sum_{k=1}^{m} b_k \right) P_i$$

where a_j are transitions into State i, and b_k are transitions leaving State i as shown in the above diagram..

Example Diagram with Associated DE Equations:
Two black boxes start operation at the same time. Box A has failure rate a and Box B has failure rate b. Successful system operation requires that Box A or Box B or both be functional. Find P_f .

The State Diagram at the right shows a system with 4 states namely (1), (2), (3), and (4). Let P1 denote the probability of the system being in State 1, i.e. the "Full Up" state (no failures). Let P2 denote the probability of the system being in State 2 etc. The arrowed lines represent transitions from one state to another. The a and b of this example represent transition rates i.e. the rate of which one state is transitioning to another. Then according to the above "general" DE equation:

Markov State Diagram



$$dP1/dt = -(a+b)P1, \quad dP2/dt = aP1 - bP2, \quad dP3/dt = bP1 - aP3, \quad dP4/dt = bP2 + aP3$$

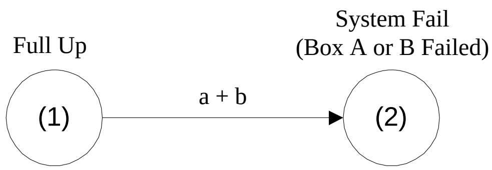
Note: States may not have any input transitions or output transitions as shown by States (1) and (4) above.

Another quantitative analysis method, known as the State Sequence Method, allows the analyst more insight into the logic flow and mathematics of the problem. The following examples utilize the DE Method to illustrate how state equations are derived, and employ the State Sequence Method to give the reader more insight into how solutions are derived (shows the Markov process flow).

2 Components in Series (Combinatorial):

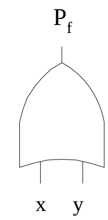
Two black boxes start operation at the same time. Box A has failure rate a and Box B has failure rate b. Successful system operation requires that both Box A and Box B be functional. Find P_f = Probability of System Failure.
Note: Full Up State = all devices operating, (n) = State Number, P(n) = Pn = Probability of being in State (n)

Markov Model



$$P(1) = e^{-(a+b)t}$$
$$P(2) = 1 - e^{-(a+b)t}$$

FTA Approach



$$x = 1 - e^{-at} \qquad y = 1 - e^{-bt}$$
$$P_f = x + y - xy = 1 - e^{-(a+b)t}$$

Markov Method of Solution using Simultaneous Differential Equations and Laplace Transforms

Assumes P(1) = P1 = 1 and P(2) = P2 = 0 when t = 0.

$$dP1/dt = -(a+b)P1 \Rightarrow L(dP1/dt) = L(-(a+b)P1) \Rightarrow s\,L(P1) - P1(0) = -(a+b)\,L(P1) \Rightarrow$$
$$(s+a+b)\,L(P1) = 1 \Rightarrow L(P1) = 1/(s+a+b) \Rightarrow P1 = e^{-(a+b)t}$$
$$P2 = 1 - P1 \Rightarrow P2 = 1 - e^{-(a+b)t}$$

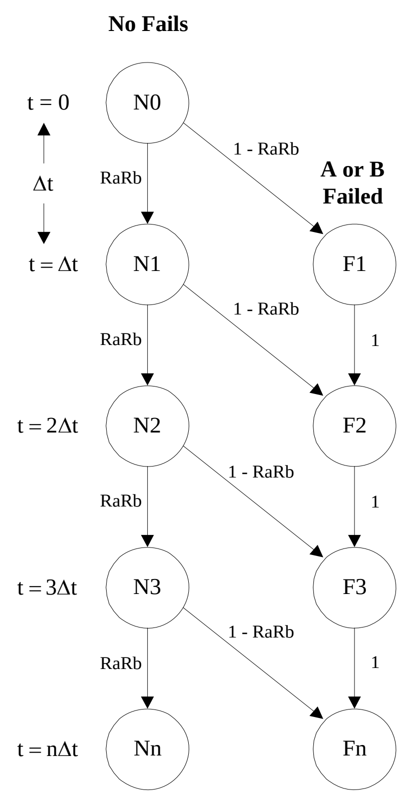
State Sequence Method

When solving state probability problems using simultaneous DE methods, the logic flow of the original problem easily gets lost due to the seemingly unrelated efforts required to calculate a set of DE solutions. The State Sequence method used for solving Markov problems illustrates the solution process clearly, and allows the analyst more insight into the problem's nature. In addition this method offers some easy to construct computer algorithms that can be used to generate solutions.

Refer to the State Sequence Diagram to the right. Let a = failure rate of A, and b = failure rate of B. Therefore $Ra = e^{-a\Delta t}$ and $Rb = e^{-b\Delta t}$ = Reliability of A and B for one fixed time interval $\Delta t \Rightarrow$

$$P(F1) = 1 - RaRb$$
$$P(F2) = 1\cdot(1 - RaRb) + RaRb\cdot(1 - RaRb) = (1 + RaRb)(1 - RaRb)$$
$$P(F3) = (1 + RaRb + Ra^2Rb^2)(1 - RaRb)$$
$$\vdots$$
$$P(Fn) = (1 + RaRb + Ra^2Rb^2 + \ldots + Ra^{n-1}Rb^{n-1})(1 - RaRb)$$
$$\Rightarrow P(Fn) = \frac{(1 - Ra^nRb^n)}{(1 - RaRb)}(1 - RaRb) = 1 - Ra^nRb^n \quad **$$
$$= 1 - (e^{-a\Delta t})^n(e^{-b\Delta t})^n = 1 - (e^{-an\Delta t})(e^{-bn\Delta t})$$
$$= 1 - (e^{-at})(e^{-bt}) = 1 - e^{-(a+b)t}$$

State Sequence Diagram

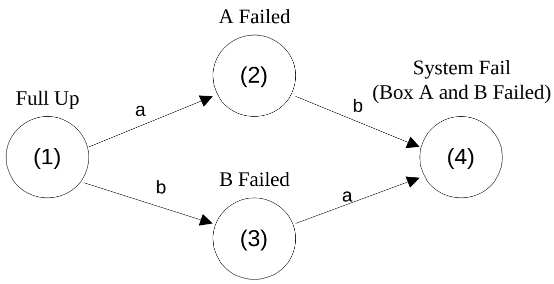


Notes :

1. The above methods assume constant failure rate devices i.e. devices that exhibit a constant probability of success (or failure) in a fixed time interval Δt.
2. P(Fn) = Sum of all transition linked Paths connecting N0 to Fn. Path = Product of all linked transitions.
3. The Sequence Method can end at **, and the expression $1 - Ra^nRb^n$ used for computer quantitative calculation. However, the problem was continued to illustrate how probability equations can be derived using this method.

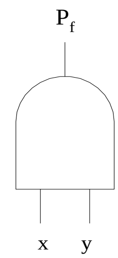
2 Components in Parallel (Combinatorial):
Two black boxes start operation at the same time. Box A has failure rate a and Box B has failure rate b. Successful system operation requires that Box A or Box B or both be functional. Find P_f .

Markov Model



$$P(1) = e^{-(a+b)t}$$
$$P(2) = e^{-bt} - e^{-(a+b)t}$$
$$P(3) = e^{-at} - e^{-(a+b)t}$$
$$P(4) = (1 - e^{-at})(1 - e^{-bt})$$

FTA Approach



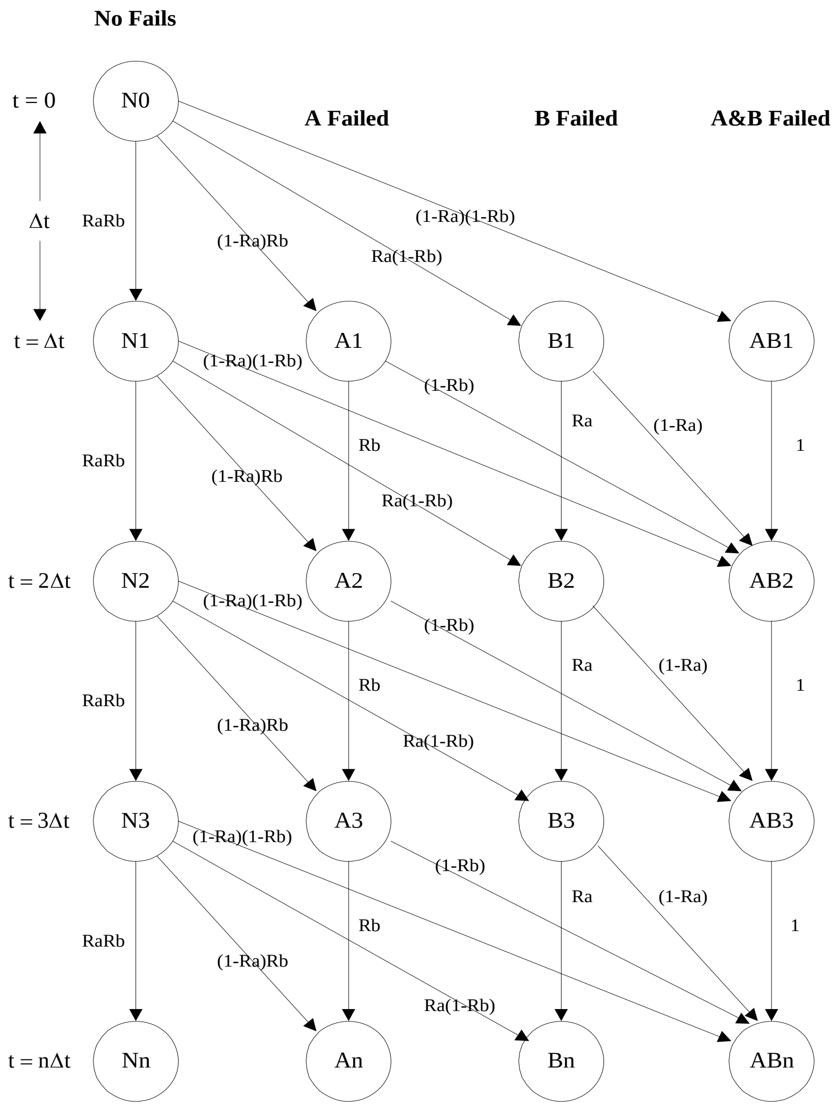
$$x = 1 - e^{-at} \qquad y = 1 - e^{-bt}$$
$$P_f = xy = (1 - e^{-at})(1 - e^{-bt})$$

Markov Method of Solution using Laplace Transforms
From the Markov Diagram above, the 4 differential equations are easily read as follows:

$dP1/dt = -(a+b)P1$, $dP2/dt = aP1 - bP2$, $dP3/dt = bP1 - aP3$, $dP4/dt = bP2 + aP3$

$dP1/dt = -(a+b)P1 \Rightarrow L(dP1/dt) = L(-(a+b)P1) \Rightarrow s\,L(P1) - 1 = -(a+b)\,L(P1) \Rightarrow$
$(s+a+b)\,L(P1) = 1 \Rightarrow L(P1) = 1/(s+a+b) \Rightarrow P1 = e^{-(a+b)t}$
$dP2/dt = aP1 - bP2 \Rightarrow L(dP2/dt) = L(aP1 - bP2) \Rightarrow sL(P2) = aL(P1) - bL(P2) \Rightarrow$
$(s+b)L(P2) = a/(s+a+b) \Rightarrow L(P2) = a/(s+b)(s+a+b) \Rightarrow P2 = e^{-bt} - e^{-(a+b)t}$
$dP3/dt = bP1 - aP3 \Rightarrow L(dP3/dt) = L(bP1 - aP3) \Rightarrow sL(P3) = bL(P1) - aL(P3) \Rightarrow$
$(s+a)L(P3) = bL(P1) \Rightarrow L(P3) = b/(s+a)(s+a+b) \Rightarrow P3 = e^{-at} - e^{-(a+b)t}$
$P4 = 1 - P1 - P2 - P3 = 1 - e^{-at} - e^{-bt} + e^{-(a+b)t} = (1 - e^{-at})(1 - e^{-bt})$

2 Components in Parallel State Sequence Diagram



Derivation of the "A failed State" Equation (2 in Parallel)

Recall $Ra = e^{-a\Delta t}$ and $Rb = e^{-b\Delta t}$ = Reliability of A and B for one fixed time interval Δt

$$\text{Let } z = (1 - Ra)$$

$$P(A1) = Rb \cdot z$$

$$P(A2) = RbP(A1) + RaRb^{2}z = (Rb^{2} + RaRb^{2})z = (Ra + 1)Rb^{2}z$$

$$P(A3) = RbP(A2) + Ra^{2}Rb^{3}z = (Ra + 1)Rb^{3}z + Ra^{2}Rb^{3}z = (Ra^{2} + Ra + 1)Rb^{3}z$$

$$P(A4) = RbP(A3) + Ra^{3}Rb^{4}z = (Ra^{2} + Ra + 1)Rb^{4}z + Ra^{3}Rb^{4}z = (Ra^{3} + Ra^{2} + Ra + 1)Rb^{4}z$$

$$\vdots$$

$$P(An) = RbP(An - 1) + Ra^{n-1}Rb^{n}z = (Ra^{n-2} + Ra^{n-3} + \cdots + Ra + 1)Rb^{n}z + Ra^{n-1}Rb^{n}z =$$

$$(Ra^{n-1} + Ra^{n-2} + \cdots + Ra + 1)Rb^{n}z = \frac{(1 - Ra^{n})}{(1 - Ra)} \cdot Rb^{n}z = (1 - Ra^{n})Rb^{n} = (1 - e^{-an\Delta t})(e^{-bn\Delta t}) \Rightarrow$$

$$P(An) = (1 - e^{-at})(e^{-bt})$$

Derivation of the "B failed State" Equation (2 in Parallel)

Let $z = (1 - Rb)$

$P(B1) = Ra \cdot z$

$P(B2) = RaP(B1) + Ra^2 Rbz = (Ra^2 + Ra^2 Rb)z = (Rb + 1)Ra^2 z$

$P(B3) = RaP(B2) + Ra^3 Rb^2 z = (Rb + 1)Ra^3 z + Ra^3 Rb^2 z = (Rb^2 + Rb + 1)Ra^3 z$

$P(B4) = RaP(B3) + Ra^4 Rb^3 z = (Rb^2 + Rb + 1)Ra^4 z + Ra^4 Rb^3 z = (Rb^3 + Rb^2 + Rb + 1)Ra^4 z$

$\vdots$

$P(Bn) = RaP(Bn - 1) + Rb^{n-1} Ra^n z = (Rb^{n-2} + Rb^{n-3} + \cdots + Rb + 1)Ra^n z + Rb^{n-1} Ra^n z =$

$(Rb^{n-1} + Rb^{n-2} + \cdots + Rb + 1)Ra^n z = \dfrac{(1 - Rb^n)}{(1 - Rb)} \cdot Ra^n z = (1 - Rb^n)Ra^n = (1 - e^{-bt})(e^{-at})$

Derivation of the "A and B failed State" Equation (2 in Parallel)

Let $z = (1 - Ra)(1 - Rb)$

$P(AB1) = z$

$P(AB2) = P(AB1) + (RaRb + Ra + Rb)z = (RaRb + Ra + Rb + 1)z = (Ra + 1)(Rb + 1)z$

$P(AB3) = P(AB2) + (Ra^2 Rb^2 + Ra^2 Rb + RaRb^2 + Ra^2 + Rb^2)z =$

$(Ra^2 Rb^2 + Ra^2 Rb + RaRb^2 + Ra^2 + Rb^2 + RaRb + Ra + Rb + 1)z = (Ra^2 + Ra + 1)(Rb^2 + Rb + 1)z$

$\vdots$

$P(ABn) = (Ra^{n-1} + Ra^{n-2} + \cdots + Ra + 1)(Rb^{n-1} + Rb^{n-2} + \cdots + Rb + 1)z =$

$\dfrac{(1 - Ra^n)}{(1 - Ra)} \cdot \dfrac{(1 - Rb^n)}{(1 - Rb)} \cdot z = \dfrac{(1 - Ra^n)}{(1 - Ra)} \cdot \dfrac{(1 - Rb^n)}{(1 - Rb)} \cdot (1 - Ra)(1 - Rb) = (1 - Ra^n)(1 - Rb^n) =$

$= (1 - e^{-an\Delta t})(1 - e^{-bn\Delta t}) = (1 - e^{-at})(1 - e^{-bt})$

Derivation of the "No fail State" Equation (2 in Parallel)

$P(N1) = RaRb$

$P(N2) = Ra^2 Rb^2$

$\vdots$

$P(Nn) = Ra^n Rb^n = (e^{-an\Delta t})(e^{-bn\Delta t}) = (e^{-at})(e^{-bt}) = e^{-(a+b)t}$

Preliminary details for the State Sequence Method for solving Non-Combinatorial problems

1. $x \xrightarrow{\text{limit}} 0 \quad e^{-x} = 1$
2. Recall a commonly known rule that is utilized quite often:

   $x \xrightarrow{\text{limit}} 0 \quad 1-e^{-x} = x$

   Proof

   $e^{-x} = 1 - x + x^2/2! - x^3/3! + x^4/4! - \cdots \Rightarrow 1-e^{-x} = x - x^2/2! + x^3/3! - x^4/4! + \cdots \Rightarrow$

   $1-e^{-x} = x(1 - x/2! + x^2/3! - x^3/4! - \cdots)$ and $x \xrightarrow{\text{limit}} 0 \; (1 - x/2! + x^2/3! - x^3/4! - \cdots) = 1 \Rightarrow$

   $x \xrightarrow{\text{limit}} 0 \quad 1-e^{-x} = x(1) = x$

   In other words, when x is very small, the term $1-e^{-x}$ can be replaced by x (or visa versa).

3. $\Delta t \to 0 \Rightarrow a\Delta t$ and $b\Delta t \to 0$ since a and b are constants $\Rightarrow$ as $\Delta t \to 0$, $1-e^{-a\Delta t} \to a\Delta t$ and $1-e^{-b\Delta t} \to b\Delta t$.

4. $a, b \xrightarrow{\text{limit}} 0 \quad e^{-a} + e^{-b} - 1 = e^{-(a+b)}$

   Proof

   $e^{-a} = 1 - a + a^2/2! - a^3/3! + a^4/4! - \cdots \Rightarrow 1-e^{-a} = a - a^2/2! + a^3/3! - a^4/4! + \cdots \Rightarrow$

   $e^{-b} = 1 - b + b^2/2! - b^3/3! + b^4/4! - \cdots \Rightarrow 1-e^{-b} = b - b^2/2! + b^3/3! - b^4/4! + \cdots \Rightarrow$

   (1) $e^{-a} + e^{-b} - 1 = 1 - (a+b) + \dfrac{a^2+b^2}{2!} - \dfrac{a^3+b^3}{3!} + \dfrac{a^4+b^4}{4!} \cdots$

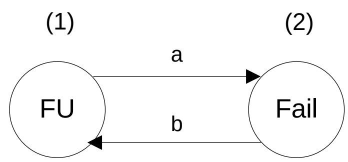
   $e^{-(a+b)} = 1 - (a+b) + \dfrac{(a+b)^2}{2!} - \dfrac{(a+b)^3}{3!} + \dfrac{(a+b)^4}{4!} \cdots \Rightarrow$

   (2) $e^{-(a+b)} = 1 - (a+b) + \dfrac{a^2+2ab+b^2}{2!} - \dfrac{a^3+3a^2b+3ab^2+b^3}{3!} + \cdots$

Now as $a \to 0$ and $b \to 0$ powers and products of a and b become insignificant compared to (a+b) and can be treated as zeroes $\Rightarrow$ (1) $\approx$ (2)

Component with Repair (Non-Combinatorial):
A black box has failure rate a and an average repair rate b. Immediately upon detection of a failure, the box goes into a repair process and put back online. Calculate the probabilities of States (1) and (2).
Note: Full Up State = device operating, (n) = State Number, P(n) = Pn = Probability of being in State (n)

Markov Model **FTA Approach**



$$P(1) = \frac{b}{a+b} + \frac{a}{a+b} e^{-(a+b)t} \qquad P(2) = \frac{a}{a+b} - \frac{a}{a+b} e^{-(a+b)t}$$

Markov Method of Solution using Simultaneous Differential Equations and Laplace Transforms
Assumes P(1) = P1 = 1 and P(2) = P2 = 0 when t = 0.

$$dP1/dt = -aP1 + bP2 \Rightarrow L(dP1/dt) = L(-aP1 + bP2) \Rightarrow s\,L(P1) - 1 = -a\,L(P1) + bL(P2)$$

$$dP2/dt = aP1 - bP2 \Rightarrow L(dP2/dt) = L(aP1 - bP2) \Rightarrow s\,L(P2) - 0 = a\,L(P1) - bL(P2)$$

$$\because [(s+a)L(P1) - 1]/b = L(P2) \text{ and } L(P2) = a\,L(P1)/(s+b) \Rightarrow [(s+a)L(P1) - 1]/b = a\,L(P1)/(s+b) \Rightarrow$$

$$(s+a)L(P1) - ab\,L(P1)/(s+b) = 1 \Rightarrow L(P1) = 1/[(s+a) - ab/(s+b)] = (s+b)/[(s+a)(s+b) - ab] \Rightarrow$$

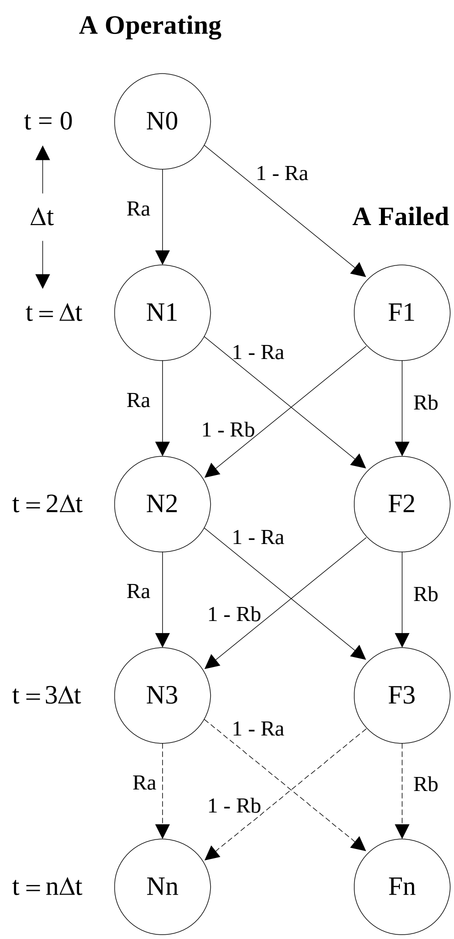
$$L(P1) = (s+b)/s(s+a+b) = 1/(s+a+b) + b/s(s+a+b) \Rightarrow P1 = L^{-1}(1/(s+a+b)) + L^{-1}(b/s(s+a+b)) \Rightarrow$$

$$P1 = e^{-(a+b)t} + \frac{b}{a+b} - \frac{b}{a+b} e^{-(a+b)t} = \frac{b}{a+b} + \frac{a}{a+b} e^{-(a+b)t}$$

$$P2 = 1 - P1 = 1 - \left(\frac{b}{a+b} + \frac{a}{a+b} e^{-(a+b)t} \right) = \frac{a}{a+b} - \frac{a}{a+b} e^{-(a+b)t}$$

Component with Repair State Sequence Diagram

Refer to the State Sequence Diagram below. Let a = failure rate of the device, and b = repair rate. Let Ra = $e^{-a\Delta t}$ and Rb = $e^{-b\Delta t}$. Then Ra = Reliability of A, and 1–Rb = probability of a repair for one fixed time interval Δt.



Component with Repair State Sequence Method

For the sake of simplifying notation, let Ni = Prob(Ni) = Probability of being in State Ni, and similarly let Fi = Prob(Fi) = Probability of being in State Fi.

Referring to the 1 Component with Repair State Sequence Diagram:

$N1 = Ra$

$N2 = Ra \cdot N1 + (1-Rb)F1 = Ra \cdot N1 + (1-Rb)(1-N1)$

$N3 = Ra \cdot N2 + (1-Rb)(1-N2)$

$\vdots$

$Nn = Ra \cdot N_{n-1} + (1-Rb)(1-N_{n-1}) \Rightarrow$

$Nn = (Ra + Rb-1)N_{n-1} + (1-Rb)$

Let $k = Ra + Rb-1$ and $m = 1-Rb \Rightarrow Nn = k \cdot N_{n-1} + m \Rightarrow$

$N2 = k \cdot N1 + m = k \cdot Ra + m$

$N3 = k^2Ra + km + m = k^2Ra + m(k + 1)$

$N4 = k^3Ra + k^2m + km + m = k^3Ra + m(k^2 + k + 1)$

$\vdots$

$Nn = k^{n-1}Ra + m(k^{n-2} + k^{n-3} + \cdots + k + 1) \Rightarrow$

$$Nn = k^{n-1}Ra + m\frac{k^{n-1}-1}{k-1} = k^{n-1}Ra + \frac{m}{k-1}(k^{n-1}-1) = (Ra + \frac{m}{k-1})k^{n-1} - \frac{m}{k-1} = (Ra - \frac{m}{1-k})k^{n-1} + \frac{m}{1-k}$$

Calculating for m, 1–k, Ra, and k^{n-1} as $\Delta t \to 0$

$m = 1-Rb = 1-e^{-b\Delta t} = b\Delta t$ as $\Delta t \to 0$

$1-k = (1-Ra) + (1-Rb) = (1-e^{-a\Delta t}) + (1-e^{-b\Delta t}) = a\Delta t + b\Delta t$ as $\Delta t \to 0$

$Ra = e^{-a\Delta t} = 1$ as $\Delta t \to 0$

$k = Ra + Rb - 1 = e^{-a\Delta t} + e^{-b\Delta t} - 1 = e^{-(a\Delta t + b\Delta t)} = e^{-(a+b)\Delta t}$ as $\Delta t \to 0 \Rightarrow$

$k^{n-1} = e^{-(a+b)(n-1)\Delta t}$ as $\Delta t \to 0 \Rightarrow$

$$\because \; Nn = (Ra - \frac{m}{1-k})k^{n-1} + \frac{m}{1-k} = (1-\frac{b\Delta t}{a\Delta t + b\Delta t})e^{-(a+b)(n-1)\Delta t} + \frac{b\Delta t}{a\Delta t + b\Delta t} \Rightarrow$$

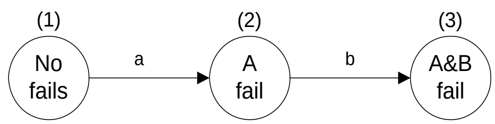
$$Nn = \left(1-\frac{b}{a+b}\right)e^{-(a+b)(n\Delta t-\Delta t)} + \frac{b}{a+b} = \frac{a}{a+b}e^{-(a+b)t} + \frac{b}{a+b} \text{ and}$$

$$Fn = 1-Nn = 1-\left(\frac{a}{a+b}e^{-(a+b)t} + \frac{b}{a+b}\right) = \frac{a}{a+b} - \frac{a}{a+b}e^{-(a+b)t} \text{ as } \Delta t \to 0$$

Note: The above method assumed a constant failure rate device i.e. a device that exhibits a constant probability of success (or failure) in a fixed time interval Δt.

2 Components Standby Redundant (Non-combinatorial):
Box A has failure rate a and Box B has failure rate b. Box A is powered on while Box B remains off. Immediately upon detection of Box A failure, Box B is powered on. Calculate the probability that both boxes fail.

Markov Model

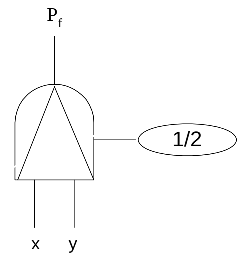


$$P(1) = e^{-a\,t}$$

$$P(2) = \frac{a}{a-b}\left(e^{-b\,t} - e^{-a\,t}\right)$$

$$P(3) = \frac{b}{a-b}\left(e^{-a\,t}\right) - \frac{a}{a-b}\left(e^{-b\,t}\right) + 1$$

FTA Approach



$$x = 1 - e^{-at} \qquad y = 1 - e^{-bt}$$

$$P_f = 1/2\ xy = 1/2\ (1 - e^{-at})(1 - e^{-bt})$$

Markov Method of Solution using DEs and Laplace Transforms

$dP1/dt = -aP1$, $dP2/dt = aP1 - bP2$, $dP3/dt = bP2$

$dP1/dt = -aP1 \Rightarrow L(dP1/dt) = L(-aP1) \Rightarrow s\,L(P1) - P1(0) = -a\,L(P1) \Rightarrow$

$(s + a)\,L(P1) = 1 \Rightarrow L(P1) = 1/(s + a) \Rightarrow P1 = e^{-a\,t}$

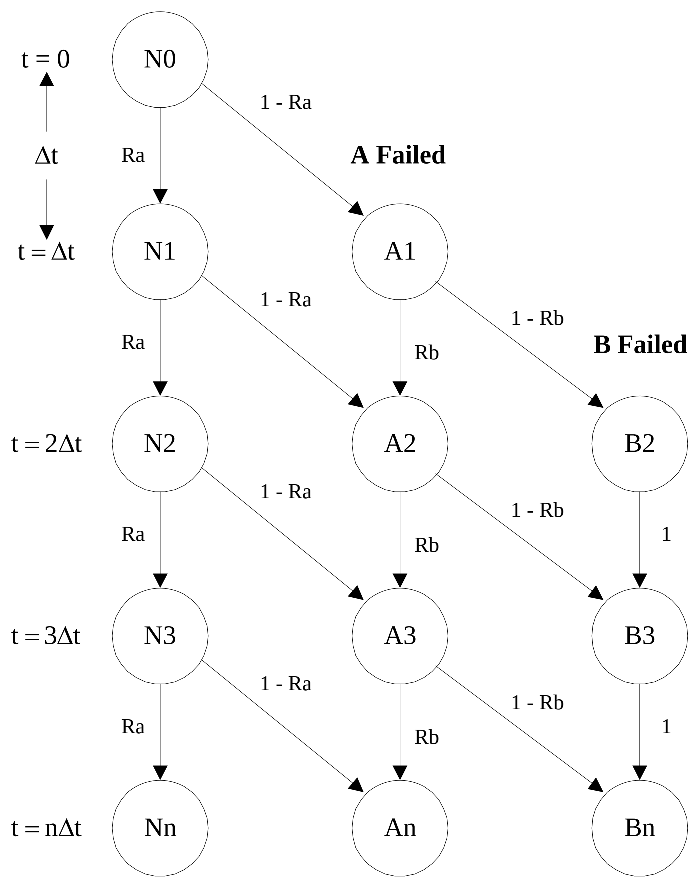
$dP2/dt = aP1 - bP2 \Rightarrow L(dP2/dt) = L(aP1 - bP2) \Rightarrow sL(P2) = aL(P1) - bL(P2) \Rightarrow$

$sL(P2) = a/(s + a) - bL(P2) \Rightarrow (s + b)L(P2) = a/(s + a) \Rightarrow L(P2) = a/(s + a)(s + b) \Rightarrow$

From Partial Fractions $a/(s + a)(s + b) = \dfrac{a/(a-b)}{s+b} - \dfrac{a/(a-b)}{s+a} \Rightarrow P2 = \dfrac{a}{a-b}e^{-bt} - \dfrac{a}{a-b}e^{-at}$

$P3 = 1 - P1 - P2 = 1 - e^{-at} - \dfrac{a}{a-b}e^{-bt} + \dfrac{a}{a-b}e^{-at} \Rightarrow P3 = 1 + \dfrac{b}{a-b}e^{-at} - \dfrac{a}{a-b}e^{-bt}$

Standby State Sequence Diagram



Derivation of the Standby "A failed State" Equation

Recall $Ra = e^{-a\Delta t}$ and $Rb = e^{-b\Delta t}$ = Reliability of A and B for one fixed time interval Δt

Let $z = (1 - Ra)$

$P(A1) = z$

$P(A2) = RbP(A1) + Raz = (Ra + Rb)z$

$$P(A3) = RbP(A2) + Ra^2 z = (Ra^2 + RaRb + Rb^2)z = \frac{Ra^3 - Rb^3}{Ra - Rb} \cdot z$$

$$P(A4) = RbP(A3) + Ra^3 z = (Ra^3 + Ra^2 Rb + RaRb^2 + Rb^3)z = \frac{Ra^4 - Rb^4}{Ra - Rb} \cdot z$$

$$\vdots$$

$$P(An) = RbP(An-1) + Ra^{n-1} z = \frac{Ra^n - Rb^n}{Ra - Rb} \cdot z = \frac{1 - Ra}{Ra - Rb}(Ra^n - Rb^n) = \frac{1 - Ra}{Rb - Ra}(Rb^n - Ra^n) =$$

$$\frac{1 - e^{-a\Delta t}}{1 + Rb - Ra - 1}(e^{-bn\Delta t} - e^{-an\Delta t}) = \frac{1 - e^{-a\Delta t}}{(1 - Ra) - (1 - Rb)}(e^{-bt} - e^{-at}) = \frac{1 - e^{-a\Delta t}}{(1 - e^{-a\Delta t}) - (1 - e^{-b\Delta t})}(e^{-bt} - e^{-at})$$

Choosing Δt very small we get $P(An) = \dfrac{a\Delta t}{a\Delta t - b\Delta t}(e^{-bt} - e^{-at}) = \dfrac{a}{a - b}(e^{-bt} - e^{-at})$

which agrees with the DE solution.

Derivation of the Standby "No fail State" Equation

N1 = Ra

N2 = Ra^2

$\vdots$

Nn = $Ra^n = (e^{-a\Delta t})^n = e^{-an\Delta t} = e^{-at}$

Derivation of the Standby "A and B failed State" Equation

We can easily solve for P3 using the fact that P3 = 1–P1–P2.

$$P3 = 1 - e^{-at} - \frac{a}{a-b}(e^{-bt} - e^{-at}) \Rightarrow P3 = 1 + \frac{b}{a-b}e^{-at} - \frac{a}{a-b}e^{-bt}$$

However, to futher illustrate the State Sequence Method:

Let $z = (1-Ra)(1-Rb)$

$$P(B2) = z = \left(\frac{Ra-Rb}{Ra-Rb}\right)z$$

$$P(B3) = P(B2) + (Ra+Rb)z = \left(\frac{Ra-Rb}{Ra-Rb} + \frac{Ra^2-Rb^2}{Ra-Rb}\right)z$$

$$(B4) = P(B3) + (Ra^2 + RaRb + Rb^2)z = \left(\frac{Ra-Rb}{Ra-Rb} + \frac{Ra^2-Rb^2}{Ra-Rb} + \frac{Ra^3-Rb^3}{Ra-Rb}\right)z$$

$$\vdots$$

$$P(Bn) = P(Bn-1) + \left(\frac{Ra^{n-1}-Rb^{n-1}}{Ra-Rb}\right)z = \left[Ra + Ra^2 + \cdots + Ra^{n-1} - (Rb + Rb^2 + \cdots + Rb^{n-1})\right]\frac{z}{Ra-Rb}$$

$$= \left(\frac{Ra-Ra^n}{1-Ra} - \frac{Rb-Rb^n}{1-Rb}\right)\frac{(1-Ra)(1-Rb)}{Ra-Rb} = \left[(1-Rb)(Ra-Ra^n) - (1-Ra)(Rb-Rb^n)\right]\frac{1}{Ra-Rb} \Rightarrow$$

$$P(Bn) = \left[(1-e^{-b\Delta t})(e^{-a\Delta t} - e^{-an\Delta t}) - (1-e^{-a\Delta t})(e^{-b\Delta t} - e^{-bn\Delta t})\right]\cdot\frac{1}{(1-e^{-b\Delta t}) - (1-e^{-a\Delta t})}$$

Choosing Δt very small we get

$$P(Bn) = \left[b\Delta t\,(e^{-a\Delta t} - e^{-at}) - a\Delta t\,(e^{-b\Delta t} - e^{-bt})\right]\cdot\frac{1}{b\Delta t - a\Delta t} =$$

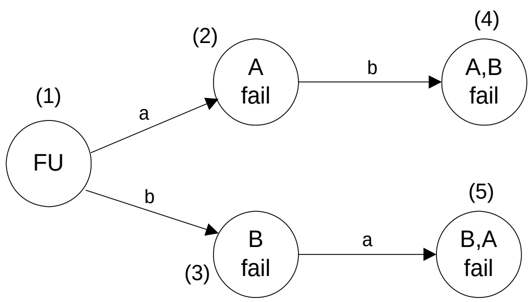
$$\left[b\,(e^{-a\Delta t} - e^{-at}) - a\,(e^{-b\Delta t} - e^{-bt})\right]\cdot\frac{1}{b-a} = \left[b\,((1-a\Delta t) - e^{-at}) - a\,((1-b\Delta t) - e^{-bt})\right]\cdot\frac{1}{b-a} =$$

$$\left[(b - ab\Delta t - be^{-at}) - ((a - ab\Delta t) - ae^{-bt})\right]\cdot\frac{1}{b-a} = \left[(b - a - be^{-at} + ae^{-bt})\right]\cdot\frac{1}{b-a} \Rightarrow$$

$$P(Bn) = 1 - \frac{b}{b-a}e^{-at} + \frac{a}{b-a}e^{-bt}$$ which agrees with the DE solution.

Required Order Factor (ROF) (Non-Combinatorial):
Two black boxes start operation at the same time. Box A has failure rate a and Box B has failure rate b. Successful system operation requires that Box A or Box B or both be functional. What is the probability that both Boxes A and B fail, and that A fails before B. Also find the probability that both Boxes fail and that B fails before A.

Markov Model



$$P(1) = e^{-(a+b)t}$$

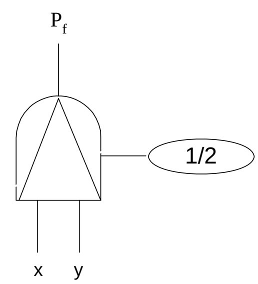
$$P(2) = e^{-bt} - e^{-(a+b)t}$$

$$P(3) = e^{-at} - e^{-(a+b)t}$$

$$P(4) = \frac{a}{a+b} + \frac{b}{a+b} e^{-(a+b)t} - e^{-bt}$$

$$P(5) = \frac{b}{a+b} + \frac{a}{a+b} e^{-(a+b)t} - e^{-at}$$

FTA Approach



$$x = 1 - e^{-at} \qquad y = 1 - e^{-bt}$$

$$P_f = 1/2\ xy = 1/2\ (1 - e^{-at})(1 - e^{-bt})$$

Markov Method of Solution using Laplace Transforms
From the Markov Diagram above, the 5 differential equations are easily read as follows:

$dP1/dt = -(a+b)P1,\quad dP2/dt = aP1 - bP2,\quad dP3/dt = bP1 - aP3,\quad dP4/dt = bP2,\quad dP5/dt = aP3$

$$dP1/dt = -(a+b)P1 \Rightarrow P1 = e^{-(a+b)t}$$

$$dP2/dt = aP1 - bP2 \Rightarrow L(P2) = a/(s+b)(s+a+b) \Rightarrow P2 = e^{-bt} - e^{-(a+b)t}$$

$$dP3/dt = bP1 - aP3 \Rightarrow L(P3) = b/(s+a)(s+a+b) \Rightarrow P3 = e^{-at} - e^{-(a+b)t}$$

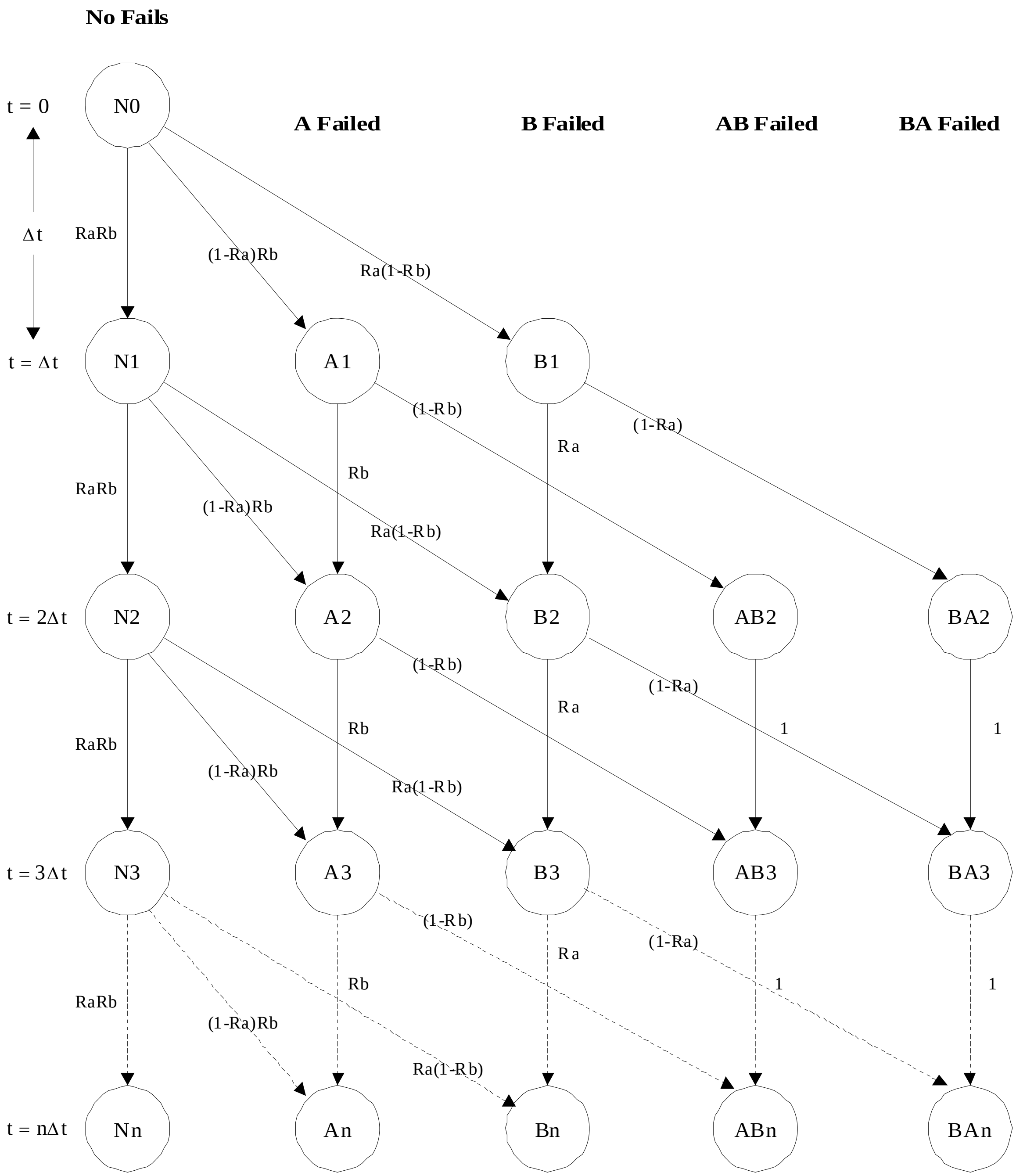
$$dP4/dt = bP2 \Rightarrow L(dP4/dt) = L(bP2) \Rightarrow sL(P4) = bL(P2) = ab/(s+b)(s+a+b) \Rightarrow$$

$$L(P4) = ab/s(s+b)(s+a+b) \Rightarrow P(4) = \frac{a}{a+b} + \frac{b}{a+b} e^{-(a+b)t} - e^{-bt}$$

$$dP5/dt = aP3 \Rightarrow L(dP5/dt) = L(aP3) \Rightarrow sL(P5) = aL(P3) = ab/(s+a)(s+a+b) \Rightarrow$$

$$L(P5) = ab/s(s+a)(s+a+b) \Rightarrow P(5) = \frac{b}{a+b} + \frac{a}{a+b} e^{-(a+b)t} - e^{-at}$$

ROF State Sequence Diagram (Version 1)



ROF (Constant Failure Rate Version 1)

$Ra = e^{-a\Delta t}, Rb = e^{-b\Delta t}$, $N_1 = N_0RaRb$, $N_2 = N_0Ra^2Rb^2$, $N_n = N_0Ra^nRb^n$

$A_1 = N_0 (1-Ra)Rb$
$A_2 = A_1Rb + N_1 (1-Ra)Rb = N_0 (1-Ra)Rb^2 + N_1 (1-Ra)Rb$
$A_2 = (N_0Rb + N1)(1-Ra)Rb = (N_0Rb + N_0RaRb)(1-Ra)Rb = N_0Rb^2(1 + Ra)(1-Ra)$
$A_3 = A_2 Rb + N_2(1-Ra)Rb = N_0Rb^3(1 + Ra)(1-Ra) + N_0Ra^2Rb^3(1-Ra)$
$A_3 = N_0Rb^3(1- Ra)[1+ Ra + Ra^2] = N_0Rb^3(1- Ra^3)$

$A_4 = A_3 Rb + N_3(1-Ra)Rb$
$A_4 = N_0Rb^4(1- Ra)[1+ Ra + Ra^2] + N_0Ra^3Rb^4(1-Ra)$
$A_4 = N_0Rb^4(1- Ra)[1+ Ra + Ra^2 + Ra^3] = N_0Rb^4(1- Ra^4)$
...
$A_n = N_0Rb^n (1 - Ra^n)$

$AB2 = A1(1-Rb) = N_0 (1-Ra)(1-Rb)Rb$

$AB3 = AB2 + A2(1-Rb)$
$AB3 = N_0Rb(1-Ra)(1-Rb) + N_0Rb^2(1- Ra^2) (1-Rb)$
$AB3 = N_0Rb(1-Ra)(1-Rb)[1 + Rb(1+ Ra)] = N_0Rb(1-Rb)[(1-Ra) + Rb(1- Ra^2)]$

$AB4 = AB3 + A3(1-Rb)$
$AB4 = N_0Rb(1-Rb)[(1-Ra) + Rb(1- Ra^2)] + N_0Rb^3(1- Ra^3) (1-Rb)$
$AB4 = N_0Rb(1-Rb)[(1-Ra) + Rb(1- Ra^2) + Rb^2(1- Ra^3)]$

$AB5 = AB4 + A4(1-Rb)$
$AB5 = N_0Rb(1-Rb)[(1-Ra) + Rb(1- Ra^2) + Rb^2(1- Ra^3)] + N_0Rb^4(1- Ra^4)(1-Rb)$
$AB5 = N_0Rb(1-Rb)[(1-Ra) + Rb(1- Ra^2) + Rb^2(1- Ra^3) + Rb^3(1- Ra^4)]$
... =>

$$AB_n = N_0Rb(1-Rb)\sum_{i=1}^{n-1} Rb^{i-1}(1-Ra^i) = N_0e^{-b\Delta t}(1-e^{-b\Delta t})\sum_{i=1}^{n-1} e^{-b(i-1)\Delta t}(1-e^{-ai\Delta t}) =>$$

$$AB_n \approx N_0b\Delta t\sum_{i=1}^{n-1} e^{-b(i-1)\Delta t}(1-e^{-ai\Delta t}) = N_0b\sum_{i=1}^{n-1} e^{-b(i-1)\Delta t}(1-e^{-ai\Delta t})\Delta t$$

To convert discrete AB_n to contiuous, let $\Delta t \to 0$, let $dx = \Delta t$, and let $i\Delta t = idx = x$, then

$$AB_n \approx N_0b\sum_{i=1}^{n-1} e^{-b(i-1)\Delta t}(1-e^{-ai\Delta t})\Delta t = N_0b\int_0^t e^{-bx}(1-e^{-ax})dx \Rightarrow$$

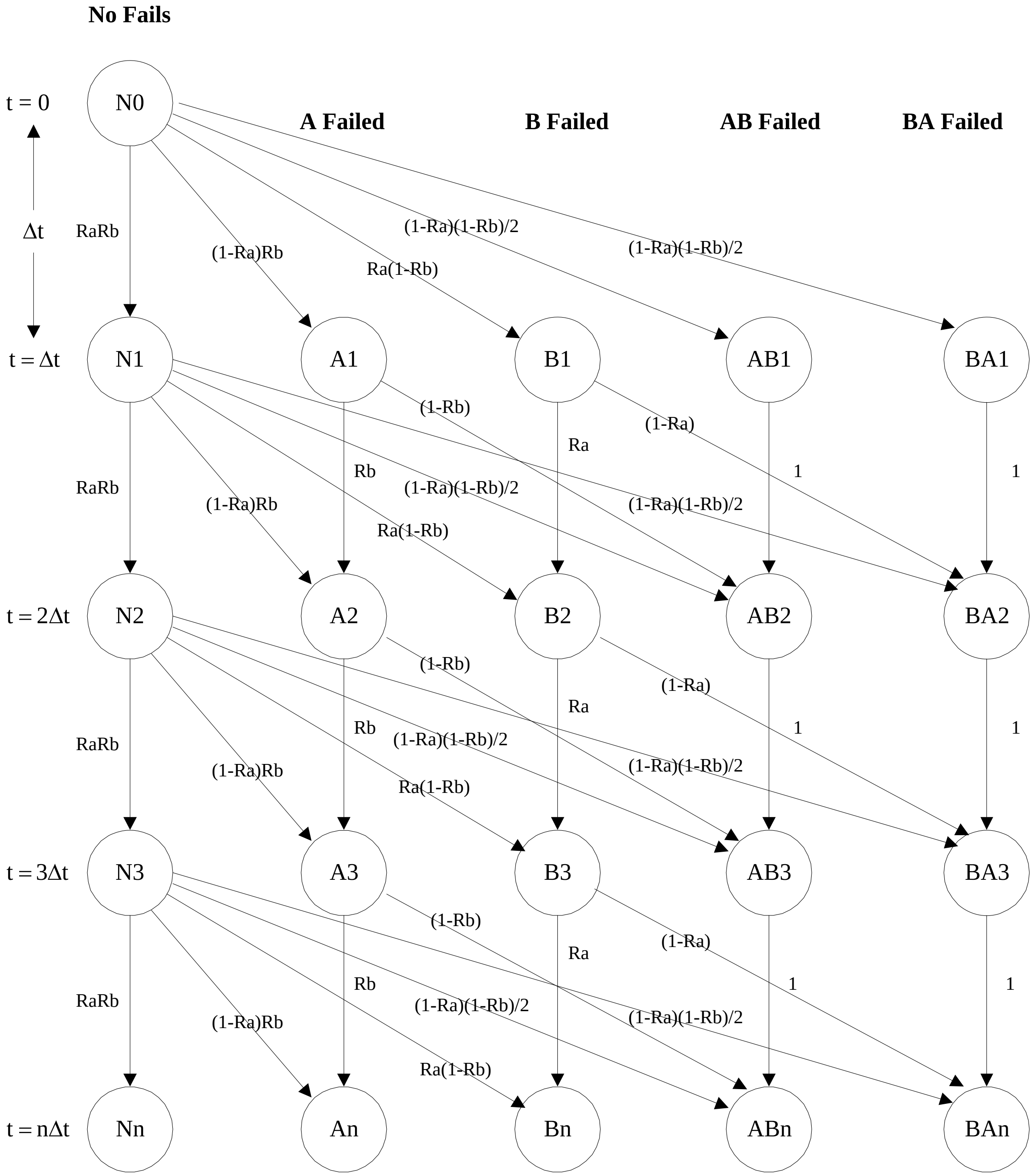
$$AB_n \approx N_0[\frac{a}{a+b} + \frac{b}{a+b}e^{-(a+b)t} - e^{-bt}]$$

In terms of pdfs:

$$AB_n \approx N_0b\int_0^t e^{-bx}(1-e^{-ax})dx = N_0\int_0^t be^{-bx}(1-e^{-ax})dx = N_0\int_0^t pdfb(x)\int_0^x pdfa(z)dz\,dx \Rightarrow$$

$$pdf(AB_n) = N_0\, pdfb(x)\int_0^x pdfa(z)dz$$

ROF State Sequence Diagram Version 2



ROF State Sequence Equations Version 2 (A fails before B state)

Let $z = (1 - Ra)(1 - Rb)$

$P(AB1) = z/2$

$P(AB2) = P(AB1) + (RaRb/2 + Rb)z$

$P(AB3) = P(AB2) + (Ra^2Rb^2 / 2 + RaRb^2 + Rb^2)z$

$P(AB4) = P(AB3) + (Ra^3Rb^3 / 2 + Ra^2Rb^3 + RaRb^3 + Rb^3)z$

$\vdots$

$$P(ABn) = P(ABn-1) + \left[Ra^{n-1}Rb^{n-1} / 2 + Rb^{n-1}(Ra^{n-2} + Ra^{n-3} + \cdots + Ra + 1)\right]z \Rightarrow$$

$$P(ABn) = P(ABn-1) + \left[Ra^{n-1}Rb^{n-1} / 2 + Rb^{n-1}\left(\frac{1 - Ra^{n-1}}{1 - Ra}\right)\right]z$$

Expanding each of the above equations yields:

$$P(AB1) = \frac{1}{2}z$$

$$P(AB2) = \left[\frac{1 - Ra^2Rb^2}{2(1 - RaRb)} + \frac{Rb(1 - Ra)}{1 - Ra}\right]z$$

$$P(AB3) = \left[\frac{1 - Ra^3Rb^3}{2(1 - RaRb)} + \frac{Rb(1 - Ra)}{1 - Ra} + \frac{Rb^2(1 - Ra^2)}{1 - Ra}\right]z$$

$$P(AB4) = \left[\frac{1 - Ra^4Rb^4}{2(1 - RaRb)} + \frac{Rb(1 - Ra)}{1 - Ra} + \frac{Rb^2(1 - Ra^2)}{1 - Ra} + \frac{Rb^3(1 - Ra^3)}{1 - Ra}\right]z$$

$\vdots$

$$P(ABn) = \left[\frac{1 - Ra^nRb^n}{2(1 - RaRb)} + \frac{Rb(1 - Ra) + Rb^2(1 - Ra^2) + \cdots + Rb^{n-1}(1 - Ra^{n-1})}{1 - Ra}\right]z =$$

$$\left[\frac{1 - Ra^nRb^n}{2(1 - RaRb)} + \frac{(Rb^{n-1} + Rb^{n-2} + \cdots + Rb + 1) - (Ra^{n-1}Rb^{n-1} + Ra^{n-2}Rb^{n-2} + \cdots + RaRb + 1)}{1 - Ra}\right]z =$$

$$\left[\frac{1 - Ra^nRb^n}{2(1 - RaRb)} + \frac{1 - Rb^n}{(1 - Ra)(1 - Rb)} - \frac{(1 - Ra^nRb^n)}{(1 - Ra)(1 - RaRb)}\right](1 - Ra)(1 - Rb) =$$

$$\frac{(1 - Ra)(1 - Rb)}{2} \cdot \frac{1 - Ra^nRb^n}{(1 - RaRb)} + 1 - Rb^n - (1 - Rb)\frac{1 - Ra^nRb^n}{1 - RaRb} = \left[\frac{(1 - Ra)(1 - Rb)}{2} - (1 - Rb)\right]\frac{1 - Ra^nRb^n}{(1 - RaRb)} + 1 - Rb^n =$$

$$\left[\frac{(-1 - Ra)}{2}(1 - Rb)\right]\frac{1 - Ra^nRb^n}{(1 - RaRb)} + 1 - Rb^n \qquad \text{Choosing } \Delta t \text{ very small we get}$$

$$P(ABn) = \frac{-2}{2} \cdot b\Delta t \cdot \frac{1 - e^{-(a+b)n\Delta t}}{(a + b)\Delta t} + 1 - e^{-bn\Delta t} = -b \cdot \frac{1 - e^{-(a+b)t}}{a + b} + 1 - e^{-bt} = \frac{a}{a + b} + \frac{b}{a + b}e^{-(a+b)t} - e^{-bt}$$

Note that by symmetry, P(BAn) is obtained simply by substituting a for b and b for a to get

$$P(BAn) = \frac{b}{a + b} + \frac{a}{a + b}e^{-(a+b)t} - e^{-at}$$

ROF State Sequence Equations Version 2a (A fails before B state)

Assume $N0 = 1$ then

$A1 = (1-Ra)Rb$

$A2 = A1Rb + N1(1-Ra)Rb = (1-Ra)Rb^2 + Ra(1-Ra)Rb^2 = (1+Ra)(1-Ra)Rb^2 = (1-Ra^2)Rb^2$

$A3 = A2Rb + N2(1-Ra)Rb = (1+Ra)(1-Ra)Rb^3 + Ra^2(1-Ra)Rb^3 = (1+Ra+Ra^2)(1-Ra)Rb^3 = (1-Ra^3)Rb^3$

...... $\Rightarrow$

$A_n = (1-Ra^n)Rb^n$

Let $z = (1-Ra)(1-Rb)$

$AB1 = z/2$

$AB2 = AB1 + A1(1-Rb) + N1z/2 = z/2 + (1-Ra)(1-Rb)Rb + RaRbz/2 = (1+RaRb)z/2 + Rbz$

$AB3 = AB2 + A2(1-Rb) + N2z/2 = (1+RaRb)z/2 + Rbz + (1-Ra)(1+Ra)Rb^2(1-Rb) + Ra^2Rb^2z/2 \Rightarrow$

$AB3 = (1+RaRb + Ra^2Rb^2)z/2 + Rbz + (1+Ra)Rb^2z = (1+RaRb + Ra^2Rb^2)z/2 + [Rb + (1+Ra)Rb^2]z$

$AB4 = AB3 + A3(1-Rb) + N3z/2 \Rightarrow$

$AB4 = (1+RaRb + Ra^2Rb^2)z/2 + Rbz + Rb^2(1+Ra)z + (1-Ra)(1+Ra+Ra^2)Rb^3(1-Rb) + Ra^3Rb^3z/2 \Rightarrow$

$AB4 = (1+RaRb + Ra^2Rb^2 + Ra^3Rb^3)z/2 + Rbz + Rb^2(1+Ra)z + (1+Ra+Ra^2)Rb^3z \Rightarrow$

$AB4 = (1+RaRb + Ra^2Rb^2 + Ra^3Rb^3)z/2 + [Rb + (1+Ra)Rb^2 + (1+Ra+Ra^2)Rb^3]z \Rightarrow$

$\vdots$

$$ABn = \left(\frac{1-Ra^nRb^n}{1-RaRb}\right)z/2 + z\sum_{i=1}^{n-1}\left(\frac{1-Ra^i}{1-Ra}\right)Rb^i = \left(\frac{1-Ra^nRb^n}{1-RaRb}\right)z/2 + (1-Rb)\sum_{i=1}^{n-1}\left(1-Ra^i\right)Rb^i \Rightarrow$$

$$ABn = \frac{1}{2}\left(\frac{1-Ra^nRb^n}{1-RaRb}\right)(1-Ra)(1-Rb) + (1-Rb)\sum_{i=1}^{n-1}Rb^i\left(1-Ra^i\right) \Rightarrow$$

$$ABn = \frac{1}{2}\left(\frac{1-e^{-n(a+b)\Delta t}}{1-e^{-(a+b)\Delta t}}\right)(1-e^{-a\Delta t})(1-e^{-b\Delta t}) + (1-e^{-b\Delta t})\sum_{i=1}^{n-1}e^{-bi\Delta t}\left(1-e^{-ai\Delta t}\right)$$

Let $\Delta t = dx$ approach 0 and let $x = i\Delta t = idx$ then

$$\text{Limit } ABn = \frac{1}{2}\left(\frac{n(a+b)\Delta t}{(a+b)\Delta t}\right)ab\Delta t^2 + b\sum_{i=1}^{n-1}e^{-bx}\left(1-e^{-ax}\right)dx \Rightarrow$$

$$\text{Limit } ABn = \frac{nab}{2}\Delta t^2 + b\sum_{i=1}^{n-1}e^{-bx}\left(1-e^{-ax}\right)dx = b\sum_{i=1}^{n-1}e^{-bx}\left(1-e^{-ax}\right)dx \Rightarrow$$

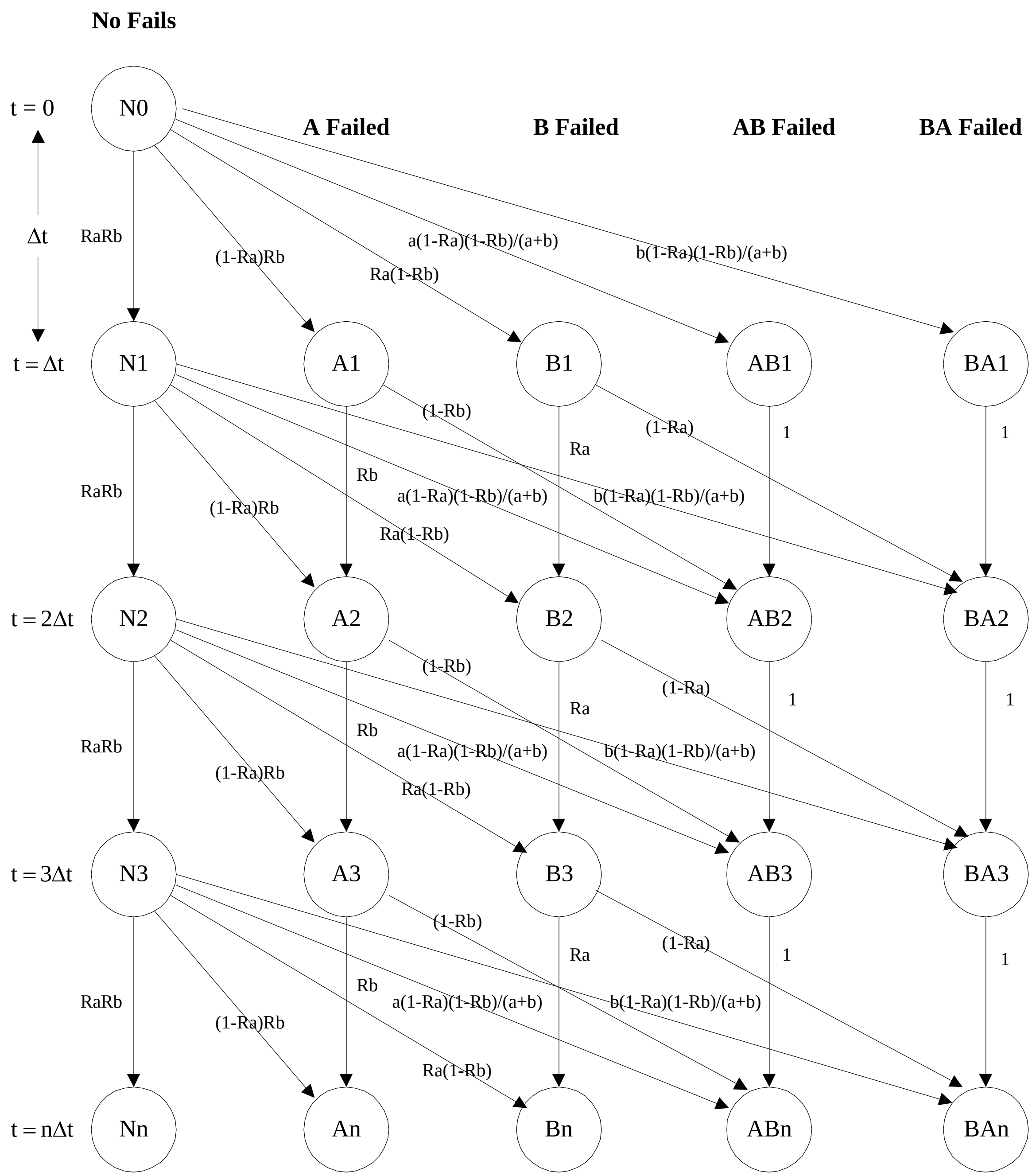
$$\text{Limit } ABn = b\int_0^t e^{-bx}\left(1-e^{-ax}\right)dx = \frac{a}{a+b} + \frac{b}{a+b}e^{-(a+b)t} - e^{-bt}$$

Note : The probability of a transition from an N State (No fail State) to an AB or BA State (both fail State) is

$P(N \to \text{Both Fail}) = k(1-Ra)(1-Rb) = k(1-e^{-a\Delta t})(1-e^{-b\Delta t})$ for some constant k.

Now the limit of $P(N \to \text{Both Fail})$ as $\Delta t \to 0 = k \cdot a\Delta t \cdot b\Delta t = kab\Delta t^2 = 0$. In other words, when Δt is very small, all the transitions from $N \to AB$ and $N \to BA$ can be disregarded.

ROF State Sequence Diagram Version 3



ROF State Sequence Equations Version 3 (A fails before B state)

Assume $N0 = 1$ then

$A1 = (1-Ra)Rb$

$A2 = A1Rb + N1(1-Ra)Rb = (1-Ra)Rb^2 + Ra(1-Ra)Rb^2 = (1+Ra)(1-Ra)Rb^2 = (1-Ra^2)Rb^2$

$A3 = A2Rb + N2(1-Ra)Rb = (1+Ra)(1-Ra)Rb^3 + Ra^2(1-Ra)Rb^3 = (1+Ra+Ra^2)(1-Ra)Rb^3 = (1-Ra^3)Rb^3$

...... $\Rightarrow$

$A_n = (1-Ra^n)Rb^n$

Let $z = (1-Ra)(1-Rb)$ and $k = a(1-Ra)(1-Rb)/(a+b)$

$AB1 = k$

$AB2 = AB1 + A1(1-Rb) + N1k = (1+N1)k + A1(1-Rb) = (1+RaRb)k + Rbz$

$AB3 = AB2 + A2(1-Rb) + N2k = (1+RaRb)k + Rbz + (1-Ra)(1+Ra)Rb^2(1-Rb) + Ra^2Rb^2k \Rightarrow$

$AB3 = (1+RaRb+Ra^2Rb^2)k + Rbz + (1+Ra)Rb^2z = (1+RaRb+Ra^2Rb^2)k + [Rb + (1+Ra)Rb^2]z$

$AB4 = AB3 + A3(1-Rb) + N3k \Rightarrow$

$AB4 = (1+RaRb+Ra^2Rb^2)k + Rbz + Rb^2(1+Ra)z + (1-Ra)(1+Ra+Ra^2)Rb^3(1-Rb) + Ra^3Rb^3k \Rightarrow$

$AB4 = (1+RaRb+Ra^2Rb^2+Ra^3Rb^3)k + Rbz + Rb^2(1+Ra)z + (1+Ra+Ra^2)Rb^3z \Rightarrow$

$AB4 = (1+RaRb+Ra^2Rb^2+Ra^3Rb^3)k + [Rb + (1+Ra)Rb^2 + (1+Ra+Ra^2)Rb^3]z \Rightarrow$

$\vdots$

$$ABn = \left(\frac{1-Ra^nRb^n}{1-RaRb}\right)k + z\sum_{i=1}^{n-1}\left(\frac{1-Ra^i}{1-Ra}\right)Rb^i = \left(\frac{1-Ra^nRb^n}{1-RaRb}\right)k + (1-Rb)\sum_{i=1}^{n-1}\left(1-Ra^i\right)Rb^i \Rightarrow$$

$$ABn = \left(\frac{a}{a+b}\right)\left(\frac{1-Ra^nRb^n}{1-RaRb}\right)(1-Ra)(1-Rb) + (1-Rb)\sum_{i=1}^{n-1}Rb^i\left(1-Ra^i\right) \Rightarrow$$

$$ABn = \left(\frac{a}{a+b}\right)\left(\frac{1-e^{-n(a+b)\Delta t}}{1-e^{-(a+b)\Delta t}}\right)(1-e^{-a\Delta t})(1-e^{-b\Delta t}) + (1-e^{-b\Delta t})\sum_{i=1}^{n-1}e^{-bi\Delta t}\left(1-e^{-ai\Delta t}\right)$$

Let $\Delta t = dx$ approach 0 and let $x = i\Delta t = idx$ then

$$\text{Limit } ABn = \left(\frac{a}{a+b}\right)\left(\frac{n(a+b)\Delta t}{(a+b)\Delta t}\right)ab\Delta t^2 + b\sum_{i=1}^{n-1}e^{-bx}\left(1-e^{-ax}\right)dx \Rightarrow$$

$$\text{Limit } ABn = \frac{na^2b}{a+b}\Delta t^2 + b\sum_{i=1}^{n-1}e^{-bx}\left(1-e^{-ax}\right)dx = b\sum_{i=1}^{n-1}e^{-bx}\left(1-e^{-ax}\right)dx \Rightarrow$$

$$\text{Limit } ABn = b\int_0^t e^{-bx}\left(1-e^{-ax}\right)dx = \frac{a}{a+b} + \frac{b}{a+b}e^{-(a+b)t} - e^{-bt}$$

$$P(AB1)=\frac{1}{2}z$$

$$P(AB2)=\left[\frac{1-Ra^2Rb^2}{2(1-RaRb)}+\frac{Rb(1-Ra)}{1-Ra}\right]z$$

$$P(AB3)=\left[\frac{1-Ra^3Rb^3}{2(1-RaRb)}+\frac{Rb(1-Ra)}{1-Ra}+\frac{Rb^2(1-Ra^2)}{1-Ra}\right]z$$

$$P(AB4)=\left[\frac{1-Ra^4Rb^4}{2(1-RaRb)}+\frac{Rb(1-Ra)}{1-Ra}+\frac{Rb^2(1-Ra^2)}{1-Ra}+\frac{Rb^3(1-Ra^3)}{1-Ra}\right]z$$

$$\vdots$$

$$P(ABn)=\left[\frac{1-Ra^nRb^n}{2(1-RaRb)}+\frac{Rb(1-Ra)+Rb^2(1-Ra^2)+\cdots+Rb^{n-1}(1-Ra^{n-1})}{1-Ra}\right]z=$$

$$\left[\frac{1-Ra^nRb^n}{2(1-RaRb)}+\frac{(Rb^{n-1}+Rb^{n-2}+\cdots+Rb+1)-(Ra^{n-1}Rb^{n-1}+Ra^{n-2}Rb^{n-2}+\cdots+RaRb+1)}{1-Ra}\right]z=$$

$$\left[\frac{1-Ra^nRb^n}{2(1-RaRb)}+\frac{1-Rb^n}{(1-Ra)(1-Rb)}-\frac{(1-Ra^nRb^n)}{(1-Ra)(1-RaRb)}\right](1-Ra)(1-Rb)=$$

$$\frac{(1-Ra)(1-Rb)}{2}\cdot\frac{1-Ra^nRb^n}{(1-RaRb)}+1-Rb^n-(1-Rb)\frac{1-Ra^nRb^n}{1-RaRb}=\left[\frac{(1-Ra)(1-Rb)}{2}-(1-Rb)\right]\frac{1-Ra^nRb^n}{(1-RaRb)}+1-Rb^n=$$

$$\left[\frac{(-1-Ra)}{2}(1-Rb)\right]\frac{1-Ra^nRb^n}{(1-RaRb)}+1-Rb^n$$ Choosing Δt very small we get

$$P(ABn)=\frac{-2}{2}\cdot b\Delta t\cdot\frac{1-e^{-(a+b)n\Delta t}}{(a+b)\Delta t}+1-e^{-bn\Delta t}=-b\cdot\frac{1-e^{-(a+b)t}}{a+b}+1-e^{-bt}=\frac{a}{a+b}+\frac{b}{a+b}e^{-(a+b)t}-e^{-bt}$$

Note that by symmetry, P(BAn) is obtained simply by substituting a for b and b for a to get

$$P(BAn)=\frac{b}{a+b}+\frac{a}{a+b}e^{-(a+b)t}-e^{-at}$$

DE / State Sequence Equation Comparisons (Constant Failure Rate type Problems)

Example	**State Probability Equations**	**Computer Expression (Algorithm)**
Series		
No Fail	$e^{-(a+b)t}$	$Ra^n Rb^n$
Fail	$1-e^{-(a+b)t}$	$1-Ra^n Rb^n$
Parallel		
No Fail	$P1 = e^{-(a+b)t}$	$P(Nn) = Ra^n Rb^n$
A Fail	$P2 = e^{-bt} - e^{-(a+b)t}$	$P(An) = Rb^n - Ra^n Rb^n$
B Fail	$P3 = e^{-at} - e^{-(a+b)t}$	$P(Bn) = Ra^n - Ra^n Rb^n$
A&B Fail	$P4 = (1-e^{-at})(1-e^{-bt})$	$P(ABn) = (1-Ra^n)(1-Rb^n)$
Repair		
No Fail	$P(1) = \frac{b}{a+b} + \frac{a}{a+b} e^{-(a+b)t}$	$P(Nn) = (Ra + Rb-1)P(N_{n-1}) + (1-Rb)$
A Fail	$P(2) = \frac{a}{a+b} - \frac{a}{a+b} e^{-(a+b)t}$	$P(An) = Rb - (Ra + Rb-1)P(N_{n-1})$
Standby		
No Fail	$P1 = e^{-at}$	$P(Nn) = Ra^n$
A Fail	$P2 = \frac{a}{a-b}(e^{-bt} - e^{-at})$	$P(An) = Rb \cdot P(A_{n-1}) + Ra^{n-1}(1-Ra)$
A&B Fail	$P3 = \frac{b}{a-b}(e^{-at}) - \frac{a}{a-b}(e^{-bt}) + 1$	$P(Bn) = P(B_{n-1}) + \left(\frac{Ra^{n-1} - Rb^{n-1}}{Ra - Rb}\right)(1-Ra)(1-Rb)$
ROF		
No Fail	$P1 = e^{-(a+b)t}$	$P(Nn) = Ra^n Rb^n$
A Fail	$P2 = e^{-bt} - e^{-(a+b)t}$	$P(An) = Rb^n - Ra^n Rb^n$
B Fail	$P3 = e^{-at} - e^{-(a+b)t}$	$P(Bn) = Ra^n - Ra^n Rb^n$
AB Fail	$P4 = \frac{a}{a+b} + \frac{b}{a+b} e^{-(a+b)t} - e^{-bt}$	$P(AB_n) = P(AB_{n-1}) + \left[Ra^{n-1}Rb^{n-1}/2 + Rb^{n-1}\left(\frac{1-Ra^{n-1}}{1-Ra}\right)\right]z$
BA Fail	$P5 = \frac{b}{a+b} + \frac{a}{a+b} e^{-(a+b)t} - e^{-at}$	$P(BA_n) = P(BA_{n-1}) + \left[Ra^{n-1}Rb^{n-1}/2 + Ra^{n-1}\left(\frac{1-Rb^{n-1}}{1-Rb}\right)\right]z$
		where $Ra = e^{-a\Delta t}$, $Rb = e^{-b\Delta t}$, and $z = (1-Ra)(1-Rb)$

Note that with respect to the combinatorial type problems, the equations and associated computer expressions are quickly interchangeable. Note also that this is not the case with the non-combinatorial types.

Visual Basic Code (Constant Failure Rates)

Standby

```
Public Sub StateSeq()
'Standby State Sequence Model    F(2,i) = P(2, idt)    A2,B2 current values, A1,B1, previous values
'Code for P(An) = Rb·P(An–1)+Ra^(n–1)(1–Ra)
EVALUATE ("dt=1"): A2 = 0: FX(2, 0) = 1 - Exp(A2)
A1 = A2: A2 = EVALUATE("-0.01*dt"): B2 = EVALUATE("-0.015*dt")
FX(2, 1) = Exp(B2) * FX(2, 0) + Exp(A1) * (1 - Exp(A2 - A1))
dt = 1
Nextdt: dt = dt + 1: If dt > 120 Then Exit Sub
EVALUATE ("dt=" & dt): A1 = A2: b1 = B2: A2 = EVALUATE("-0.01*dt"): B2 = EVALUATE("-0.015*dt")
FX(2, dt) = Exp(-0.015) * FX(2, dt - 1) + Exp(A1) * (1 - Exp(-0.01))
GoTo Nextdt
End Sub
```

Repair

```
Public Sub StateSeq()
'Repair State Sequence Model, F(2,i) = P(1, idt),  A2,B2 current values, A1,B1, previous values
'Code for P(Nn) = (Ra + Rb-1)P(Nn-1)+ (1-Rb)
EVALUATE ("dt=1"): FX(2, 0) = 1
A1 = A2: b1 = B2: A2 = EVALUATE("exp(-0.01*dt)"): B2 = EVALUATE("exp(-0.015*dt)")
FX(2, 1) = (A2 + B2 - 1) * FX(2, 0) + 1 - B2
dt = 1
Nextdt: dt = dt + 1: If dt > 120 Then Exit Sub
EVALUATE ("dt=" & dt): A1 = A2: b1 = B2: A2 = EVALUATE("exp(-0.01)"): EVALUATE ("exp(-0.015)")
FX(2, dt) = (A2 + B2 - 1) * FX(2, dt - 1) + 1 - B2
GoTo Nextdt
End Sub
```

ROF

```
Public Sub StateSeq()
'ROF State Sequence Model, F(2,i) = P(4, idt),  A2,B2 current values, A1,B1, previous values
'Code for P(ABn) = P(ABn-1)+{Ran-1Rbn-1/2+ Rbn-1[(1- Ran-1)/( 1-Ra)]}(1-Ra)(1-Rb)
EVALUATE ("dt=1"): A2 = 0: B2 = 0: FX(2, 0) = 0
A1 = A2: b1 = B2: A2 = EVALUATE("-0.01*dt"): B2 = EVALUATE("-0.015*dt")
FX(2, 1) = FX(2, 0) + (Exp(A2) * Exp(B2) / 2 + Exp(B2) * (1 - Exp(A2)) / (1 - Exp(-0.01))) * (1 - Exp(-0.01)) * (1
- Exp(-0.015))
dt = 1
Nextdt: dt = dt + 1: If dt > 120 Then Exit Sub
EVALUATE ("dt=" & dt): A1 = A2: b1 = B2: A2 = EVALUATE("-0.01*dt"): B2 = EVALUATE("-0.015*dt")
FX(2, dt) = FX(2, dt - 1) + (Exp(A2) * Exp(B2) / 2 + Exp(B2) * (1 - Exp(A2)) / (1 - Exp(-0.01))) * (1 - Exp(-0.01))
* (1 - Exp(-0.015))
GoTo Nextdt
End Sub
```

Concluding Notes

1. Although Markov techniques can be utilized for both combinatorial and non-combinatorial type problems, the analyst should stick with FTA when dealing with combinatorial types, and Markov when dealing with non-combinatorial type problems.
2. Although the qualitative methods shown above can also be used for analysis of non-constant failure rate components (mechanical devices), the quantitative methods shown are limited to constant failure rate components.
3. It is suggested that the analyst utilize "Markov" computer programs when performing a quantitative analysis. The methods described above can become very exhaustive when the number of states gets large. These methods were illustrated simply to have the reader obtain a better understanding and insight into Markov techniques.
4. Several computer programs are available for solving non-combinatorial problems. Most programs utilize matrix algebra techniques, and output numeric values as opposed to equations for probability evaluation of each state. Be aware that these programs will have limitations such as number of input states and program execution speed. However this limitation keeps getting smaller with each advance in computer technology.