Chapter 2

Basics of Reliability

Definition of Reliability:
Given an electrical, mechanical, or electro-mechanical device is in operation.
To be completed

Reliability Formulas		
Term	**Basic Formula**	**Notes**
f(x), PDF	f(x) depends on distribution. See following pages. $f(x) = \frac{d}{dx}F(x)$	f(x) = PDF = probability density function
P_f, F(t)	$P_f = F(t) = \int_0^t f(x)dx$	P_f = F(t) = Probability of Failure
P_s, R(t)	$P_s = 1 - F(t) = 1 - \int_0^t f(x)dx = e^{-\int_0^t \lambda(x)dx}$	R(t) = Reliability = P_s = Probability of Success
λ(t)	$\lambda(t) = \frac{\frac{d}{dt}(1 - R(t))}{R(t)} = -\frac{d}{dt}[\ln(R(t))]$	λ(t) = Failure Rate = Hazard Rate
MTTF	$MTTF = \int_0^\infty R(t)\,dt = \int_0^\infty t \cdot f(t)\,dt$	MTTF = Mean Time to Fail
MFR	$MFR = \frac{1}{MTTF}$	MFR = Mean Failure Rate
P_f Series System	$P_f(sys) = 1 - e^{-\left(\sum_{i=1}^{n}\int_0^t \lambda_i(x)dx\right)} = 1 - \prod_{i=1}^{n} R_i(t)$	1. n components in series / parallel 2. $\lambda_i(t)$ = hazard rate of ith component 3. component failures must be independent 4. $R_i(t)$ = Rel of ith component 5. $F_i(t)$ = P_f of ith component
P_f Parallel System	$P_f(sys) = \prod_{i=1}^{n}\left[1 - e^{\left(-\int_0^t \lambda_i(x)dx\right)}\right] = \prod_{i=1}^{n} F_i(t)$	
P_f Standby System (2 devices)	$P_f(sys) = \int_0^t \int_0^x f_1(z)\, f_2(x-z)dz\,dx$ $= \int_0^t F_1(t-x)\, f_2(x)dx$ $= \int_0^t F_2(t-x)\, f_1(x)dx$	1. One component in "cold" standby of another component in operation. 2. f_i = PDF, F_i = probability of component failure.

Exponential Distribution		
Term	**Basic Formula**	**Notes**
f(t), PDF	$f(t) = a\,e^{-at}$	f(t) = PDF = probability density function
P_f , F(t)	$P_f = 1 - e^{-at}$	P_f = F(t) = Probability of Failure
P_s , R(t)	$R(t) = P_s = e^{-at}$	R(t) = Reliability = P_s = Probability of Success = $1 - P_f$
h(t)	$h(t) = \frac{f(t)}{R(t)} = a$	h(t) = Failure Rate = Hazard Rate
MTTF	$MTTF = \int_0^{\infty} e^{-at} = \frac{1}{a}$	MTTF = Mean Time to Fail
MFR	a	MFR = Mean Failure Rate
P_f Series System	$P_f\,(sys) = 1 - e^{-\left(\sum_{i=1}^{n} a_i\right)t}$	1. n components in series / parallel 2. a_i = constant hazard rate of ith component 3. component failures must be independent
P_f Parallel System	$P_f\,(sys) = \prod_{i=1}^{n} \left(1 - e^{-a_i t}\right)$	
P_f Standby System (2 devices)	$P_f\,(sys) = 1 + \frac{b}{a-b} e^{-at} - \frac{a}{a-b} e^{-bt}$	a = failure rate of device 1 b = failure rate of device 2

Normal Distribution		
Term	**Basic Formula**	**Notes**
f(x), PDF	$f(x)=\frac{1}{s\sqrt{2\pi}}e^{\left[-\frac{1}{2}\left(\frac{x-u}{s}\right)^2\right]}$	1. f(x) = PDF = probability density function 2. u = mean, s = sigma (constants)
P_f, F(t)	$P_f = F(t) = \int_{-\infty}^{t} f(x)dx$	P_f = F(t) = Probability of Failure
P_s, R(t)	$P_s = R(t) = 1 - F(t) = 1 - \int_{-\infty}^{t} f(x)dx$	R = Reliability = P_s = Probability of Success = 1 – P_f
h(t)	h(t) = f(t)/R(t)	h(t) = Failure Rate = Hazard Rate
MTTF	u	MTTF = Mean Time to Fail
MFR	$\frac{1}{u}$	MFR = Mean Failure Rate
P_f Series System	$P_f(sys) = 1 - e^{-\left(\sum_{i=1}^{n}\int_0^t h_i(x)dx\right)}$	1. n components in series / parallel 2. $h_i(t)$ = hazard rate of ith component (component failures must be independent)
P_f Parallel System	$P_f(sys) = \prod_{i=1}^{n}\left[1-e^{\left(-\int_0^t h_i(x)dx\right)}\right]$	
P_f Standby System (n devices)	$u = u_1 + u_2 + \cdots + u_n$ $s = \sqrt{s_1^2 + s_2^2 + \cdots + s_n^2}$ $k = k_1 + k_2 + \cdots + k_n$	1. 1 active component, n –1 on standby 2. u_i = mean of ith component 3. s_i = sigma 4. k_i = hours (miles) logged 5. component failures must be independent
Normal P_f Approximations		
P_f 1 device	$P_f = 1 - e^{-\left[\left(\frac{t+k+b\cdot s-u}{a\cdot s}\right)^c\right]}$	1. u_i = mean of ith component 2. s_i = sigma 3. k_i = hours (miles) logged 4. a = 3.48078264 5. b = 3.130874117 6. c = 3.45721089 7. Defined for $t \geq \max(u_i - bs_i - k_i)$ 8. $P_f = 0$ for $t < \max(u_i - bs_i - k_i)$ (component failures must be independent)
P_f n devices in Series	$P_f(sys) = 1 - e^{-\left[\sum_{i=1}^{n}\left(\frac{t+k_i+b\cdot s_i-u_i}{a\cdot s_i}\right)^c\right]}$	
P_f n devices in Parallel	$P_f(sys) = \prod_{i=1}^{n}[1-e^{-\left(\frac{t+k_i+b\cdot s_i-u_i}{a\cdot s_i}\right)^c}]$	
P_f Standby System (n devices)	$P_f = 1 - e^{-\left[\left(\frac{t+k+b\cdot s-u}{a\cdot s}\right)^c\right]}$	1. $u = u_1 + u_2 + \cdots + u_n$ 2. $s = \sqrt{s_1^2 + s_2^2 + \cdots + s_n^2}$ 3. $k = k_1 + k_2 + \cdots + k_n$ 4. a, b, and c same as above

LogNormal Distribution		
Term	**Basic Formula**	**Notes**
f(x), PDF	$f(x)=\frac{1}{sx\sqrt{2\pi}}\,e^{\left[-\frac{1}{2}\left(\frac{\ln(x)-u}{s}\right)^2\right]}$	1. f(x) = PDF = probability density function 2. u = mean, s = sigma (constants)
P_f , F(t)	$P_f = F(t) = \int_{-\infty}^{t} f(x)dx$	P_f = F(t) = Probability of Failure
P_s , R(t)	$P_s = R(t) = 1 - F(t) = 1 - \int_{-\infty}^{t} f(x)dx$	R = Reliability = P_s = Probability of Success = 1 − P_f
h(t)	h(t) = f(t)/R(t)	h(t) = Failure Rate = Hazard Rate
MTTF	$e^{\left(u+s^2/2\right)}$	MTTF = Mean Time to Fail
MFR	$e^{-\left(u+s^2/2\right)}$	MFR = Mean Failure Rate
P_f Series System	$P_f(sys) = 1 - e^{-\left(\sum_{i=1}^{n}\int_0^t h_i(x)dx\right)}$	1. n components in series / parallel 2. $h_i(t)$ = hazard rate of ith component 3. component failures must be independent
P_f Parallel System	$P_f(sys) = \prod_{i=1}^{n}\left[1 - e^{\left(-\int_0^t h_i(x)dx\right)}\right]$	
P_f Standby System (2 devices)	$P_f(sys) = \int_0^t \int_0^x f_1(z)\,f_2(x-z)dz\,dx$ $= \int_0^t F_1(t-x)\,f_2(x)dx = \int_0^t F_2(t-x)\,f_1(x)dx$	1. One component in “cold” standby of another component in operation. 2. f_i = PDF, F_i = probability of component failure.
LogNormal P_f Approximations		
P_f 1 device	$P_f = 1 - e^{-\left[\left(\frac{\ln(t+k)+b\cdot s-u}{a\cdot s}\right)^c\right]}$	1. u_i = mean of ith component 2. s_i = sigma 3. k_i = hours (miles) logged 4. component failures must be independent 5. a = 3.48078264 6. b = 3.130874117 7. c = 3.45721089 8. Defined for $t \ge \max(e^{(ui - b\cdot si)} - k_i)$ 9. $P_f = 0$ for $t < \max(e^{(ui - b\cdot si)} - k_i)$
P_f n devices in Series	$P_f(sys) = 1 - e^{-\left[\sum_{i=1}^{n}\left(\frac{\ln(t+k_i)+b\cdot s_i-u_i}{a\cdot s_i}\right)^c\right]}$	
P_f n devices in Parallel	$P_f(sys) = \prod_{i=1}^{n}[1 - e^{-\left(\frac{\ln(t+k_i)+b\cdot s_i-u_i}{a\cdot s_i}\right)^c}]$	
P_f Standby System (n devices)	$P_f = 1 - e^{-\left[\left(\frac{\ln(t+k)+b\cdot s-u}{a\cdot s}\right)^c\right]}$ Check this. Do not believe true.	1. $u = \ln(e^{u_1} + e^{u_2} + \cdots + e^{u_n})$ 2. $s = \sqrt{s_1^2 + s_2^2 + \cdots + s_n^2}$ 3. $k = k_1 + k_2 + \cdots + k_n$ 4. a, b, and c same as above

Weibull Distribution		
Term	**Basic Formula**	**Notes**
f(t), PDF	$f(t) = \frac{c(t-b)^{c-1}}{a^c} e^{-\left[\left(\frac{t-b}{a}\right)^c\right]}$	1. f(t) = PDF = probability density function 2. a, b, c are parameter constants
P_f , F(t)	$P_f = F(t) = 1 - e^{-\left[\left(\frac{t-b}{a}\right)^c\right]}$	P_f = F(t) = Probability of Failure
P_s , R(t)	$P_s = R(t) = e^{-\left[\left(\frac{t-b}{a}\right)^c\right]}$	R(t) = Reliability = P_s = Probability of Success = 1 – P_f
h(t)	$h(t) = \frac{c(t-b)^{c-1}}{a^c}$	h(t) = Failure Rate = Hazard Rate
MTTF (c=1)	$MTTF = a\, e^{\frac{b}{a}}$	1. c = 1=> Exponential Distribution 2. $e^{\frac{b}{a}} = R(t) \text{ at } t = 0$
MTTF (2.5≤ c c≤ 5)	$MTTF = b + hl + a\,[-\ln(0.5)]^{\frac{1}{c}}$ Check this???	2.5≤ c ≤ 5 => Normal Distribution
MFR	$MFR = \frac{1}{MTTF}$	MFR = Mean Failure Rate
P_f Series System	$P_f(sys) = 1 - e^{-\left[\sum_{i=1}^{n}\left(\frac{t-bi}{ai}\right)^{ci}\right]}$	1. n component failures must all be independent 2. ai, bi, and ci are parameters for ith component 3. define $\left(\frac{t-bi}{ai}\right)^{ci} = 0$ for t < bi
P_f Parallel System	$P_f(sys) = \prod_{i=1}^{n}\left[1 - e^{-\left(\frac{t-bi}{ai}\right)^{ci}}\right]$	
P_f Standby System (2 devices)	$P_f(sys) = \int_0^t \int_0^x f_1(z)\, f_2(x-z)\,dz\,dx$ $= \int_0^t F_1(t-x)\, f_2(x)\,dx$ $= \int_0^t F_2(t-x)\, f_1(x)\,dx$	1. One component in "cold" standby of another component in operation. 2. f_i = PDF, F_i = probability of component failure.

Failure Rate of n items in parallel with equal λ

P_f of one item is $1-e^{-\lambda t} \Rightarrow P_f$ of n items is $(1-e^{-\lambda t})^n \Rightarrow P_s$ of parallel branch is $1-(1-e^{-\lambda t})^n$

Let $p = e^{-\lambda t}$

$$\text{Recall } (1-p)^n = \binom{n}{0}(-p)^0 + \binom{n}{1}(-p)^1 + \binom{n}{2}(-p)^2 + \cdots + \binom{n}{n}(-p)^n \Rightarrow$$

$$P_s = 1-(1-p)^n = -\binom{n}{1}(-p)^1 - \binom{n}{2}(-p)^2 - \cdots - \binom{n}{n}(-p)^n \Rightarrow$$

$$P_s = 1-(1-e^{-\lambda t})^n = -\binom{n}{1}(-e^{-\lambda t})^1 - \binom{n}{2}(-e^{-\lambda t})^2 - \cdots - \binom{n}{n}(-e^{-\lambda t})^n \Rightarrow$$

$$P_s = 1-(1-e^{-\lambda t})^n = -\binom{n}{1}(-1)^1 e^{-\lambda t} - \binom{n}{2}(-1)^2 e^{-2\lambda t} - \cdots - \binom{n}{n}(-1)^n e^{-n\lambda t}$$

$$\text{Recall also MTTF} = \int_0^\infty P_s\, dt = \int_0^\infty \left[-\binom{n}{1}(-1)^1 e^{-\lambda t} - \binom{n}{2}(-1)^2 e^{-2\lambda t} - \cdots - \binom{n}{n}(-1)^n e^{-n\lambda t} \right] dt \Rightarrow$$

$$\text{MTTF} = -\binom{n}{1}(-1)^1 \int_0^\infty e^{-\lambda t}\, dt - \binom{n}{2}(-1)^2 \int_0^\infty e^{-2\lambda t}\, dt - \cdots - \binom{n}{n}(-1)^n \int_0^\infty e^{-n\lambda t}\, dt \Rightarrow$$

$$\text{MTTF} = \frac{n}{\lambda} - \binom{n}{2}\frac{(-1)^2}{2\lambda} - \cdots - \binom{n}{n}\frac{(-1)^n}{n\lambda} = \frac{1}{\lambda}\left(n - \binom{n}{2}\frac{(-1)^2}{2} - \cdots - \binom{n}{n}\frac{(-1)^n}{n} \right)$$

$$\text{Case } n=2 \Rightarrow \text{MTTF} = \frac{2}{\lambda} - \frac{1}{2\lambda} = \frac{3}{2\lambda} \Rightarrow \text{ Failure Rate } = \frac{2}{3}\lambda$$

$$\text{Case } n=3 \Rightarrow \text{MTTF} = \frac{3}{\lambda} - \frac{3}{2\lambda} + \frac{1}{3\lambda} = \frac{11}{6\lambda} \Rightarrow \text{ Failure Rate } = \frac{6}{11}\lambda$$

$$\text{Case } n=4 \Rightarrow \text{MTTF} = \frac{4}{\lambda} - \frac{6}{2\lambda} + \frac{4}{3\lambda} - \frac{1}{4\lambda} = \frac{25}{12\lambda} \Rightarrow \text{ Failure Rate } = \frac{12}{25}\lambda$$

$$\text{Case } n=5 \Rightarrow \text{MTTF} = \frac{5}{\lambda} - \frac{10}{2\lambda} + \frac{10}{3\lambda} - \frac{5}{4\lambda} + \frac{1}{5\lambda} = \frac{137}{60\lambda} \Rightarrow \text{ Failure Rate } = \frac{60}{137}\lambda$$

Convolution

Given two devices with one in "standby" of the other, the system probability density function (PDF_{sys}) can be derived from the PDFs of the devices. If f_1 and f_2 are the PDFs of device 1 and 2 respectively then

$$PDF_{sys} = f(t) = \int_0^t f_1(x)\, f_2(t-x)dx \quad \text{Note : This integral is known as the Convolution of } f_1 \text{ and } f_2.$$

Example:
Solution of Standby Problem Using Convolution
Electrical device A has failure rate a, and electrical device B has failure rate b. Device A is powered on while device B remains off. Immediately upon detection of device A failure, device B is powered on. Calculate the probability that both devices fail i.e. system failure.

Electrical devices $\Rightarrow$ device PDFs are $f_1(x) = ae^{-ax}$ and $f_2(x) = be^{-bx}$. Then from above definition:

$$PDF_{sys} = f(t) = \int_0^t f_1(x)\, f_2(t-x)dx = \int_0^t ae^{-ax} be^{-b(t-x)}dx = ab\int_0^t e^{-ax} e^{-b(t-x)}dx =$$

$$ab\int_0^t e^{-ax} e^{bx} e^{-bt} dx = abe^{-bt}\int_0^t e^{(b-a)x} dx = \frac{ab}{b-a} e^{-bt}\left(e^{(b-a)t} - 1\right) = \frac{ab}{b-a}\left(e^{-at} - e^{-bt}\right) \Rightarrow$$

$$\boxed{pdf_{sys} = f(t) = \frac{ab}{b-a}\left(e^{-at} - e^{-bt}\right)}$$

$$\text{Now } P_f(sys) = \int_0^t pdf_{sys}dz = \int_0^t f(z)dz = \int_0^t \frac{ab}{b-a}\left(e^{-az} - e^{-bz}\right)dz = \frac{ab}{b-a}\int_0^t \left(e^{-az} - e^{-bz}\right)dz \Rightarrow$$

$$\boxed{P_f(sys) = 1 + \frac{a}{b-a}e^{-bt} - \frac{b}{b-a}e^{-at}}$$

Solution of Standby Problem Using Convolution Derived by Laplace
The PDF of the system can be also be derived using Laplace Transforms as follows:

$$pdf_{sys} = f(t) = L^{-1}\{L[f_1(t)]\, L[f_2(t)]\}$$

$$\text{where } L[f(t)] = \text{Laplace Transform of } f(t) = \int_0^\infty e^{-st} f(t)\, dt \text{ and } f(t) = L^{-1}\left[\int_0^\infty e^{-st} f(t)\, dt\right]$$

$$\Rightarrow PDF_{sys} = f(t) = L^{-1}\left\{L[ae^{-at}]\, L[be^{-bt}]\right\} = L^{-1}\left\{\frac{a}{s+a}\cdot\frac{b}{s+b}\right\} = L^{-1}\left\{\frac{ab}{(s+a)(s+b)}\right\}$$

$$\text{Using partial fraction methods } \frac{ab}{(s+a)(s+b)} = \frac{ab/(b-a)}{s+a} - \frac{ab/(b-a)}{s+b} \Rightarrow$$

$$L^{-1}\left(\frac{ab}{(s+a)(s+b)}\right) = L^{-1}\left(\frac{ab/(b-a)}{s+a}\right) - L^{-1}\left(\frac{ab/(b-a)}{s+b}\right) = \frac{ab}{b-a}e^{-at} - \frac{ab}{b-a}e^{-bt} \Rightarrow$$

$$\boxed{PDF_{sys} = f(t) = \frac{ab}{b-a}\left(e^{-at} - e^{-bt}\right)} \Rightarrow \boxed{P_f(sys) = 1 + \frac{a}{b-a}e^{-bt} - \frac{b}{b-a}e^{-at}}$$

Numerical Analysis Approach to Convolution

Theorem:
If f and g are functions such that all their derivatives exist, and if h(x) = f(x)g(x) then

$$h^n = \binom{n}{0} f^n g^0 + \binom{n}{1} f^{n-1} g^1 + \binom{n}{2} f^{n-2} g^2 + \cdots + \binom{n}{n} f^0 g^n \quad \text{where } f^0 = f,\ g^0 = g, f^n = \text{nth derivative of } f \text{ etc.}$$

Let h(x) = f(x)g(x) then

$$h(x) = h(0) + h^1(0)\,x + \frac{h^2(0)}{2!} x^2 + \frac{h^3(0)}{3!} x^3 + \cdots \text{where}$$

$h(0) = f(0)g(0)$

$h^1(0) = f^1(0)g(0) + f(0)g^1(0)$

$h^2(0) = f^2(0)g(0) + 2 f^1(0)g^1(0) + f(0)g^2(0)$

$$h^n(0) = \binom{n}{0} f^n(0)g(0) + \binom{n}{1} f^{n-1}(0)\,g^1(0) + \binom{n}{2} f^{n-2}(0)\,g^2(0) + \cdots + \binom{n}{n} f(0)\,g^n(0)$$

Note: This example illustrates an exponential distribution convolved with another exponential distribution.

$$\text{Recall PDF (Standby System)} = \int_0^t f(x)\cdot g(t-x)\,dx \quad \text{where } f(x) = \text{PDF of Device1, and } g(x) = \text{PDF of Device2}$$

$$\therefore \text{electrical Box A operating with electrical Box B on standby} \Rightarrow \text{PDF (Standby System)} = \int_0^t ae^{-ax}\cdot be^{-b(t-x)}\,dx$$

where a is the failure rate of Box A and b is the failure rate of Box B ⇒

$$\text{PDF (Standby System)} = ab\int_0^t e^{-ax}\cdot e^{-b(t-x)}\,dx \quad \text{Let } h(x) = f(x)g(x) \text{ where } f(x) = e^{-ax} \text{ and } g(x) = e^{-b(t-x)} \Rightarrow \qquad (1)$$

$f(0) = 1,\ \ f^1(0) = -a,\ \ f^2(0) = a^2,\ \cdots,\ \ f^n(0) = (-1)^n a^n$ and

$g(0) = e^{-bt},\ g^1(0) = be^{-bt},\ g^2(0) = b^2e^{-bt}, \cdots,\ g^n(0) = b^n e^{-bt} \Rightarrow$

$h(0) = e^{-bt},\ h^1(0) = be^{-bt} - ae^{-bt},\ h^2(0) = b^2e^{-bt} - 2abe^{-bt} + a^2e^{-bt}, \cdots,$

$$h^n(0) = \binom{n}{0}(-1)^n a^n e^{-bt} + \binom{n}{1}(-1)^{n-1} a^{n-1} be^{-bt} + \binom{n}{2}(-1)^{n-2} a^{n-2} b^2 e^{-bt} + \cdots + \binom{n}{n} b^n e^{-bt} \Rightarrow$$

$$h(x) = \left[1 + (b-a)x + (b^2 - 2ab + a^2)\,x^2/2! + (b^3 - 3ab^2 + 3a^2b + a^2)\,x^3/3! + \cdots\right] e^{-bt} \Rightarrow$$

$$h(x) = \left[1 + (b-a)x + (b-a)^2 x^2/2! + (b-a)^3 x^3/3! + \cdots\right] e^{-bt} = e^{(b-a)x} e^{-bt}$$

$$\text{From (1) PDF (Standby System)} = ab\int_0^t h(x)\,dx = abe^{-bt}\int_0^t e^{(b-a)x}\,dx = \frac{ab}{b-a}(e^{-at} - e^{-bt})$$