The Poisson Approximates the Binomial

Theorem:

If n is large and q is small, then $\frac{n!}{k!(n-k)!}p^{n-k}q^{k} \approx \frac{(nq)^{k}}{k!}e^{-nq}$

Proof :

$$\frac{n!}{k!(n-k)!}p^{n-k}q^{k} = \frac{n\,(n-1)(n-2)\,\cdots(n-k+1)}{k!}(1-q)^{n-k}q^{k}$$

$$\approx \frac{n^{k}}{k!}(1-q)^{n-k}q^{k} \quad \text{since n is large}$$

$$= \frac{n^{k}}{k!}\cdot\frac{(1-q)^{n}}{(1-q)^{k}}\cdot q^{k} \approx \frac{n^{k}}{k!}\cdot\frac{(1-q)^{n}}{1}\cdot q^{k} = \frac{(nq)^{k}(1-q)^{n}}{k!} \quad \text{since q is small} \quad (1)$$

now compare $(1-q)^{n}$ with e^{-nq} by expanding them out.

$$(1-q)^{n} = 1-nq + \frac{n(n-1)}{2!}q^{2} - \frac{n(n-1)(n-2)}{3!}q^{3} + \cdots$$

$$\approx 1-nq+\frac{n^{2}}{2!}q^{2}-\frac{n^{3}}{3!}q^{3}+\cdots = 1-nq+\frac{(nq)^{2}}{2!}-\frac{(nq)^{3}}{3!}+\cdots \quad \text{since n is large}$$

$$\therefore\ (1-q)^{n} \approx 1-nq+\frac{(nq)^{2}}{2!}-\frac{(nq)^{3}}{3!}+\cdots \quad (2)$$

$$e^{-nq} = 1-nq + \frac{(nq)^{2}}{2!} - \frac{(nq)^{3}}{3!}+\cdots \quad (3)$$

comparing (2) and (3) $\Rightarrow (1-q)^{n} \approx e^{-nq}$ (4)

Replacing e^{-nq} for $(1-q)^{n}$ in (1) we get $\frac{n!}{k!(n-k)!}p^{n-k}q^{k} \approx \frac{(nq)^{k}}{k!}e^{-nq}$ //