**Sum of Powers Theorem** 4/08/05

The largest order term of $S_r = 1^r + 2^r + 3^r + \cdots + n^r = \dfrac{n^{r+1}}{r+1}$ for all integers $r > 0$

Proof: (by induction and the use of the telescope method)

$$r = 1 \Rightarrow S_1 = 1 + 2 + 3 + \cdots + n = \frac{n}{2}(n+1) = \frac{n^2}{2} + \frac{n}{2} \Rightarrow \text{theorem true for } r = 1$$

Now assume true for r i.e. $S_r = \dfrac{n^{r+1}}{r+1}$ and show true for $r+1$

$$(n+1)^{r+2} = n^{r+2} + (r+2)\,n^{r+1} + k\,n^r + \text{lower order terms } \left(\text{binomial theorem with } k = \frac{(r+2)!}{2\cdot r!}\right) \Rightarrow$$

$$\begin{aligned}
(n+1)^{r+2} - (n)^{r+2} &= (r+2)\,(n)^{r+1} + k\,n^r + \text{lower order terms} \Rightarrow\\
(n)^{r+2} - (n-1)^{r+2} &= (r+2)\,(n-1)^{r+1} + k\,(n-1)^r + \text{lower order terms} \Rightarrow\\
(n-1)^{r+2} - (n-2)^{r+2} &= (r+2)\,(n-2)^{r+1} + k\,(n-2)^r + \text{lower order terms} \Rightarrow\\
(n-2)^{r+2} - (n-3)^{r+2} &= (r+2)\,(n-3)^{r+1} + k\,(n-3)^r + \text{lower order terms} \Rightarrow\\
&\vdots\\
(3)^{r+2} - (2)^{r+2} &= (r+2)\,(2)^{r+1} + k\,(2)^r + \text{lower order terms} \Rightarrow\\
(2)^{r+2} - (1)^{r+2} &= (r+2)\,(1)^{r+1} + k\,(1)^r + \text{lower order terms} \Rightarrow
\end{aligned}$$

$$(n+1)^{r+2} - (1)^{r+2} = (r+2)\sum_{i=1}^{n} i^{r+1} + k\sum_{i=1}^{n} i^r + \text{lower order terms} \Rightarrow$$

$$(r+2)\sum_{i=1}^{n} i^{r+1} = (n+1)^{r+2} - k\sum_{i=1}^{n} i^r + \text{lower order terms} \Rightarrow$$

$$(r+2)\sum_{i=1}^{n} i^{r+1} = (n)^{r+2} + (r+2)\,(n)^{r+1} + k\cdot n^r - k\sum_{i=1}^{n} i^r + \text{lower order terms} \Rightarrow$$

$$\sum_{i=1}^{n} i^{r+1} = S_{r+1} = \frac{(n)^{r+2} + (r+2)\,(n)^{r+1} + k\cdot n^r - k\sum_{i=1}^{n} i^r + \text{lower order terms}}{r+2} \Rightarrow$$

the largest order term of $S_{r+1} = 1^{r+1} + 2^{r+1} + 3^{r+1} + \cdots + n^{r+1} = \dfrac{n^{r+2}}{r+2}$ //

Corollary: $\lim\limits_{n\to\infty} S_r = 1^r + 2^r + 3^r + \cdots + n^r = \dfrac{n^{r+1}}{r+1}$