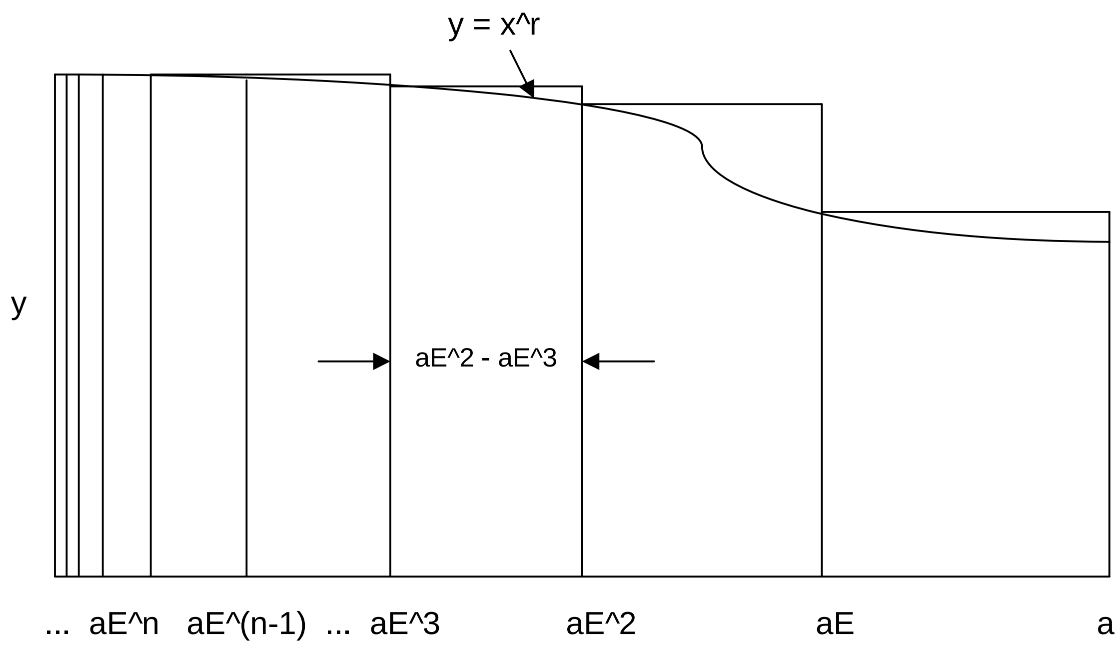
Theorem (Definite Integral by Fermat 1601-1665)
(for positive exponents)

$$\int_0^a x^r\,dx = \frac{a^{r+1}}{r+1} \text{ for real } r > 0$$



Set $E < 1$ and divide inteval [0 to a] as shown above. Note that there are an infinite number of intervals shown.
Let S = the sum of areas of all rectangles and let E approach 1.

$$S = (aE)^r(a - aE) + (aE^2)^r(aE - aE^2) + (aE^3)^r(aE^2 - aE^3) + (aE^4)^r(aE^3 - aE^4) + \cdots \Rightarrow$$

$$S = \sum_{i=1}^{\infty}(aE^i)^r(aE^{i-1} - aE^i) = a^{r+1}\sum_{i=1}^{\infty}E^{i\cdot r}(E^{i-1} - E^i) = a^{r+1}\sum_{i=1}^{\infty}E^{i\cdot r+i\text{-}1}(1-E) \Rightarrow$$

$$S = a^{r+1}(1-E)\sum_{i=1}^{\infty}E^{i\cdot r+i\text{-}1} = a^{r+1}(1-E)\left[E^r + E^{2r+1} + E^{3r+2} + E^{4r+3} + \cdots\right]$$

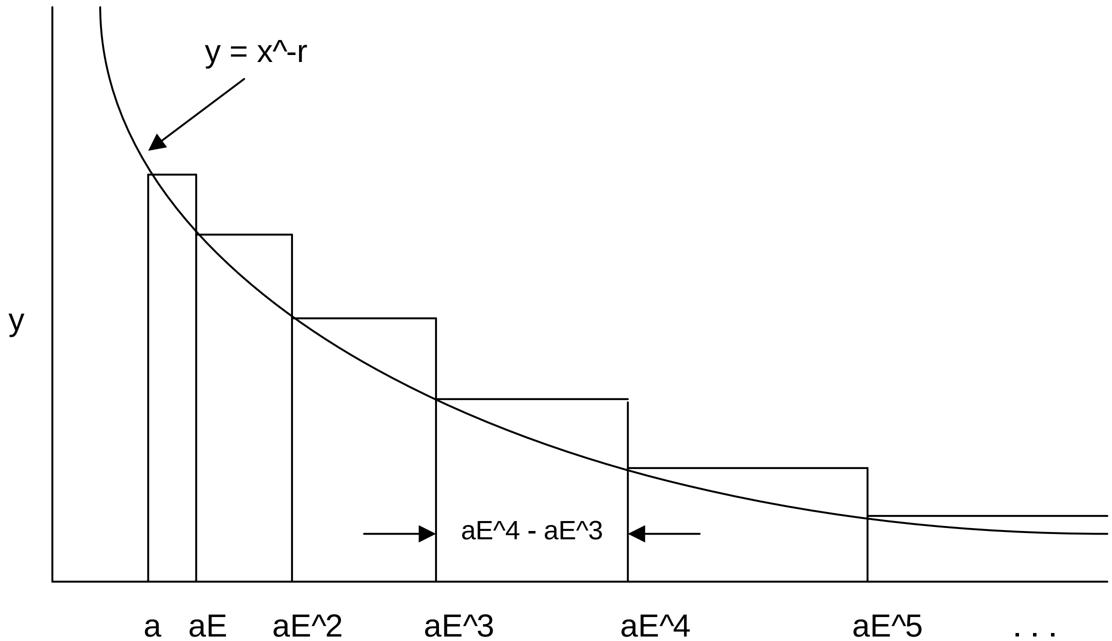
$$\text{Let } Z = E^r + E^{2r+1} + E^{3r+2} + E^{4r+3} + \cdots \Rightarrow ZE^{r+1} = E^{2r+1} + E^{3r+2} + E^{4r+3} + E^{5r+4}\cdots \Rightarrow$$

$$Z - ZE^{r+1} = E^r \Rightarrow Z = \frac{E^r}{1-E^{r+1}} \Rightarrow S = a^{r+1}(1-E)\frac{E^r}{1-E^{r+1}} = \frac{a^{r+1}E^r}{E^r + E^{r-1} + E^{r-2} + \cdots + E + 1}$$

$$\therefore \lim_{E\to 1} S = \frac{a^{r+1}}{r+1} \quad //$$

Theorem (Definite Integral by Fermat 1601-1665)
(for negative exponents)

$$\int_a^{\infty} x^{-r}dx = \frac{a^{1-r}}{r-1} \text{ for real } r > 0 \text{ and } r \neq 1$$



Set $E > 1$ and divide inteval[a to ∞] as shown above. Note that there are an infinite number of intervals shown.
Let S = the sum of areas of all rectangles and let E approach 1.

$$S = a^{-r}(aE - a) + (aE)^{-r}(aE^2 - aE) + (aE^2)^{-r}(aE^3 - aE^2) + (aE^3)^{-r}(aE^4 - aE^3) + \cdots \Rightarrow$$

$$S = \sum_{i=0}^{\infty}(aE^i)^{-r}(aE^{i+1} - aE^i) = a^{1-r}\sum_{i=0}^{\infty}E^{-i\cdot r}(E^{i+1} - E^i) = a^{1-r}\sum_{i=0}^{\infty}E^{i-i\cdot r}(E-1) \Rightarrow$$

$$S = a^{1-r}(E-1)\sum_{i=0}^{\infty}E^{i-i\cdot r} = a^{1-r}(E-1)\left[1 + E^{1-r} + E^{2-2r} + E^{3-3r} + \cdots\right]$$

$$\text{Let } Z = 1 + E^{1-r} + E^{2-2r} + E^{3-3r} + \cdots \Rightarrow Z\cdot E^{1-r} = E^{1-r} + E^{2-2r} + E^{3-3r} + E^{4-4r}\cdots \Rightarrow$$

$$Z - ZE^{1-r} = 1 \Rightarrow Z = \frac{1}{1-E^{1-r}} \Rightarrow Z = \frac{E^{r-1}}{E^{r-1}-1} \Rightarrow S = a^{1-r}(E-1)\frac{E^{r-1}}{E^{r-1}-1} \Rightarrow$$

$$S = \frac{a^{1-r}\,E^{r-1}}{E^{r-2} + E^{r-3} + E^{r-4} + \cdots + E + 1} \Rightarrow \lim_{E\to 1} S = \frac{a^{1-r}}{r-1} \quad //$$